Coverage In Heterogeneous Sensor Networks

Loukas Lazos and Radha Poovendran

Network Security Laboratory (NSL), Department of Electrical Engineering,
University of Washington, Seattle, WA,
l_lazos, radha@ee.washington.edu

Abstract— In this paper we study the problem of *coverage* in heterogeneous planar sensor networks. Coverage as a performance metric, quantifies the quality of monitoring provided by the sensor network. We formulate the problem of coverage as a set intersection problem arising in Integral Geometry, and derive analytical expressions for stochastic coverage. Our formulation allows us to consider a heterogeneous sensing model, where sensors need not have an identical sensing capability. In addition, our approach is applicable to scenarios where the sensing area of each sensor has arbitrary shape and sensors are deployed according to any distribution. We present analytical expressions only for convex sensing areas, however, our results can be generalized to non-convex areas. The validity of our expressions is verified by extensive simulations.

I. INTRODUCTION

Sensor networks are projected to have a significant impact into our everyday lives, with applications to environmental monitoring, home health care, disaster relief operations, and ambient monitoring [1]. One of the primary tasks of sensor networks is the collective monitoring of a field of interest. Sensors may monitor physical properties such as temperature, humidity, air quality, or track the motion of objects moving within the field of interest. In order for the sensor network to sufficiently monitor the entire field of interest, one needs to ensure that every point of the field is covered by at least one sensor. Furthermore, to provide the desired accuracy and robustness against node failures, many applications require that each point of the field of interest is sensed by more than one sensor. Hence, the problem of node deployment for the purpose of sensing can be viewed as a *coverage problem*, defined below.

The coverage problem is to quantify how well is the field of interest sensed by the deployment of the sensor network. The coverage problem can be studied under different objectives and constraints imposed by the applications such as, worst-case coverage [10], deterministic coverage [10], [12] or stochastic coverage [8], [10], [12], [15], [20]. The worst-case coverage problem quantifies coverage based on the parts of the field of interest that exhibit the lowest observability from the sensors [10], and is relevant in applications where a desired threshold number of sensors need to observe the field of interest. The deterministic coverage problem [10], [12] quantifies the coverage achieved by deploying sensors in a deterministic way, and is relevant in applications where one can select the positions where the sensors are placed. The stochastic coverage problem [8], [10], [12], [15], [20], on the other hand, quantifies the coverage achieved when sensors are deployed according to a distribution, and is relevant in applications where the sensors' positions cannot be selected a priori.

In this paper, we analyze the following stochastic coverage problem. Given a planar field of interest and N sensors deployed according to a known distribution, compute the fraction of the field of interest that is covered by at least k sensors ($k \geq 1$). The problem can also be rephrased as, given a field of interest and a sensor distribution, how many sensors must be deployed in order for every point in the field of interest to be covered by at least k sensors with a probability p (k-coverage problem) [20].

In this paper we make the following contributions. We formulate the problem of coverage in sensor networks as a set intersection problem. We use results from integral geometry to derive analytical expressions quantifying the coverage achieved by stochastic deployment of sensors into a planar field of interest. Compared to previous analytical results [8], [12], [20], our formulation allows us to consider a heterogeneous sensing model, where sensors need not have an identical sensing capability. In addition, our approach is applicable to scenarios where the sensing area of a sensor is not an ideal circle, but has any arbitrary shape. To the best of our knowledge, only [15] considers a heterogeneous sensing model, though only incorporating the mean value of the sensing range in the coverage computation. In addition, the formulation in [15] considers only uniformly deployed sensors. In our approach, sensors can be deployed according to any distribution. We provide formulas for k-coverage in the case of heterogeneous sensing areas, as well as the simplified forms in the case of identical sensing areas, and give an example for the computation of the number of sensors required to cover a field of interest with a pre-specified probability. Finally, we validate our theoretical expressions via simulations and show an exact match between simulation and theory.

The rest of the paper is organized as follows. In Section III we formulate the coverage problem as a set intersection problem. In Section IV we derive analytical expressions for coverage. In Section V, we validate our theoretical results via simulation. Section VI presents our conclusions.

II. RELATED WORK

In this section we describe related work to the coverage problem in wireless sensor networks. The coverage problem can be classified under different objectives and metrics. The different approaches to the coverage problem are, deterministic

or stochastic sensor deployment, homogeneous or heterogeneous sensing area, additional design constraints such as energy efficiency, minimum number of sensors that need to be deployed, or network connectivity. Based on the objective, the coverage problem formulation varies to reflect the different assumptions and objectives.

In [9], the authors study the problem of deterministic node placement in order to achieve connected coverage, that is, sense the field of interest with the minimum number of sensors, while keeping the sensor network connected. The authors assume that the sensing area of each sensor follows the unit disk model and consider sensors with identical sensing areas. The problem of connected coverage has also been recently studied in [20]. The authors provide a geometric analysis that relates coverage to connectivity and define the necessary conditions for a network covering a field of interest to be connected. The conditions for coverage and connectivity are derived based on the assumptions that the sensing area of each node is identical and circular, and the location of the nodes is known. The authors extend their algorithms for the case of probabilistic deployment, and also relax their assumptions to non-unit disk sensing areas, by approximating the real sensing area with the biggest possible circular area included in the real sensing area.

In [16] the authors study the problem of deterministic coverage under the additional constraint that each sensor must have at least k neighbors. They propose a deployment strategy that would maximize the coverage while the degree of each node is guaranteed to be at least k, under the assumption that the sensing range of the sensors is isotropic.

In [13], the authors study the problem of coverage, as a path exposure problem. Using a generic sensing model and an arbitrary sensor distribution, they propose a systematic method for discovering the minimum exposure path, that is the path along which the network exhibits the minimum *integral observability*[1]. Authors in [10], investigate the problem of best- and worst-case coverage. In their formulation of the coverage problem, given the location of the sensors and a generic sensing model where the sensing ability of each sensor diminishes with distance, the authors use Voronoi diagrams and Delaunay triangulation to compute the path that maximizes the smallest observability (best coverage) and the path that minimizes the observability by all sensors (worst coverage). In [11], the authors provide a decentralized and localized algorithm for calculating the best coverage.

Authors in [12], study the problem of stochastic coverage in large scale sensor networks. For a randomly distributed sensor network, the authors provide the fraction of the field of interest covered by k sensors, the fraction of nodes that can be removed without reducing the covered area as well as the ability of the network to detect moving objects. The results presented in [12] hold only for randomly (uniformly) deployed networks and under the assumption that the sensing

area of each sensor is identical. Furthermore, the analysis in [12] suffers from the border effects problem, illustrated in [2], [3]. The results hold asymptotically under the assumption that the field of interest expands infinitely in the plane, while the density of the sensor deployment remains constant.

In [15], the authors study the stochastic coverage problem in ad hoc networks in the presence of channel randomness. For a randomly deployed sensor network, the authors analyze the effects of shadowing and fading to the connectivity and coverage. They show that the in the case of channel randomness, the coverage problem can still be modeled after the Spatial Poisson distribution, by using *expected* size of the sensing area of sensors. While the results in [15] are applicable to heterogeneous sensor networks, they hold only for randomly deployed networks, and are impacted from the border effects problem [2], [3], as noted by the authors [15].

Compared to previous work that derives analytical coverage expressions [12], [15], [16], our formulation allows us to consider a network model where, (a) sensors can be deployed according to *any* distribution, (b) sensors can have a sensing area of *any* arbitrary shape, (c) sensors can have heterogeneous sensing areas.

III. Problem formulation & Background

In this section, we formulate the problem of coverage in heterogeneous sensor networks as a set intersection problem arising in Integral Geometry [7], [14], [17]–[19] and provide relevant background for the set intersection problem.

A. Problem Formulation

We formulate the problem of stochastic coverage as follows. Let $\mathcal{A}_0$ denote the planar field of interest we want to monitor, with area F_0 and perimeter L_0. Assume that N sensors with sensor s_i having a sensing area $\mathcal{A}_i$, $(i = 1 \ldots N)$, are deployed according to a distribution $K(\mathcal{A}_0)$ and in such a way that they sense some part of the field of interest[2]. Let F_i, L_i denote the size and the perimeter of the sensing area $\mathcal{A}_i$ of each sensor s_i, respectively. We want to calculate *the fraction of $\mathcal{A}_0$ that is sensed by at least k sensors, i.e. the fraction that is k-covered ($k \geq 1$)*. This problem is equivalent to computing the probability that a randomly selected point $P \in \mathcal{A}_0$ is sensed by at least k sensors. We map this coverage problem to the following set intersection problem. In our formulation, a set $\mathcal{S}$ is defined as a collection of points in the plane, and for the coverage problem the sets are closed regions.

Let $\mathcal{S}_0$ be a fixed closed set defined as a collection of points in the plane, and let F_0 and L_0 denote the area and perimeter of $\mathcal{S}_0$. Let N closed sets $\mathcal{S}_i$ ($i = 1 \ldots N$) of size F_i and perimeter L_i be dropped in the plane of $\mathcal{S}_0$ according to a distribution $K(\mathcal{S}_0)$ and in such a way that every set $\mathcal{S}_i$ intersects with $\mathcal{S}_0$. Compute the fraction of $\mathcal{S}_0$ where at least

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[1]The integral observability is defined as the aggregate of the time that a target was observable by sensors while traversing a sensor network.

[2]Note that for sensing, we do not require that sensors are located within the field of interest. Instead, as shown in Figure 1(a), we require that they can monitor some part of the field of interest even if they are located outside of it.
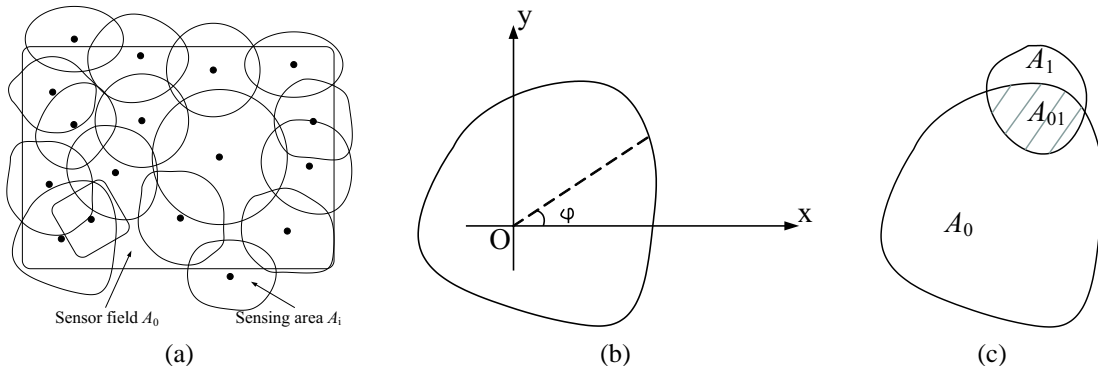
Fig. 1. (a) A heterogeneous sensor network with sensors covering the deployment region $\mathcal{A}_0$. (b) A convex set $\mathcal{A}$ and the corresponding quantities that define the kinematic density. (c) Two convex sets $\mathcal{A}_0, \mathcal{A}_1$ intersecting, and the common area $\mathcal{A}_0$.

k out of the N sets $\mathcal{S}_i$ intersect.

In the mapping of the stochastic coverage problem to the set intersection problem, the fixed closed set $\mathcal{S}_0$ corresponds to the field of intersect $\mathcal{A}_0$. The N closed sets dropped according to the distribution $K(\mathcal{S}_0)$ correspond to the sensing areas of the N sensors deployed according to the distribution $K(\mathcal{A}_0)$. By computing the fraction of the set $\mathcal{S}_0$, where at least k out of N sets $\mathcal{S}_i$ intersect, we equivalently compute the fraction of the field of interest that is k-covered[3].

The set intersection problem has been a topic of research of Integral Geometry and Geometric Probability [7], [14], [17]–[19]. Before we provide analytical coverage expressions based on our formulation, we present relevant background.

B. Background on Integral Geometry

In this section, we present relevant background on Integral Geometry that we use in Section IV for deriving analytical coverage expressions based on our formulation. Interested reader is referred to [7], [14], [17]–[19], as reference to Integral Geometry.

We first define the notion of the *kinematic density* for the group of motions of a set $\mathcal{A}$ in the plane, that is used to define a measure that quantifies the possible positions of $\mathcal{A}$, such that a specific event occurs [17]. The kinematic density expresses the differential element of motion of a set in the plane and is defined as follows.

Definition 1: **Kinematic Density**–Let $\mathcal{M}$ denote the group of motions of a set $\mathcal{A}$ in the plane. The kinematic density $d\mathcal{A}$ for the group of motions $\mathcal{M}$ in the plane for the set $\mathcal{A}$, is defined as the differential form:

$$d\mathcal{A} = dx \wedge dy \wedge d\phi, \tag{1}$$

where $\wedge$ denotes the exterior product used in exterior calculus [5], [6], (x, y) denote the Cartesian coordinates, and ϕ denotes the rotation angle of $\mathcal{A}$ with respect to the x axis of the coordinate system[4] [17].

[3]Due to their equivalence, $\mathcal{A}_0$ and $\mathcal{S}_0$ as well as the terms sensing area and set are used interchangeably in the rest of the paper.

[4]For every set $\mathcal{A}$, one can randomly choose a reference point O, based on which all translations and rotation motions of $\mathcal{A}$ are defined.

In figure 1(b), we show a set $\mathcal{S}$, a randomly selected reference point $O \in S$, and the axis of a coordinate system. All rotations and translations for the set $\mathcal{S}$ are defined with respect to the reference point O. Integrating the kinematic density of a set $\mathcal{A}$ over a group of motions $\mathcal{M}$ in the plane, yields a measure for the set of motions $\mathcal{M}$, which is called the kinematic measure [17], defined below.

Definition 2: **Kinematic measure**–The kinematic measure m of a set of motions $\mathcal{M}$ in the plane is defined by the integral of the kinematic density $d\mathcal{A}$ over $\mathcal{M}$:

$$m = \int_{\mathcal{M}} d\mathcal{A}. \tag{2}$$

By measuring the motions of a set in the plane, we quantify the space of all possible positions of the set that correspond to that motion. The quotient of the measure of any random motion path Z over the measure of all possible motions $\mathcal{M}$ in the plane, yields the probability $p(Z)$ for that random motion path Z to occur:

$$p(Z) = \frac{m(Z)}{m(\mathcal{M})}. \tag{3}$$

The kinematic measure allows us to compute the geometric probability for a specific set configuration to occur, as depicted in (3). Equation (3), is used in our formulation to derive the fraction of the field of interest covered by a sensor deployment, as it is illustrated in the following section.

IV. COVERAGE IN HETEROGENEOUS SENSOR NETWORKS

In this section, we derive analytical expressions for coverage by analyzing the coverage problem as a set intersection problem. We first illustrate the coverage computation when a single sensor is randomly deployed to monitor the field of interest, by studying the intersection of two sets in the plane. We then compute coverage when the sensor is deployed according to a distribution $K(\mathcal{A}_0)$. We extend our expressions to the general case where N sensors are deployed at random. We compute the fraction of the field of interest covered by exactly k sensors in the case of heterogeneous sensing areas and simplify the formula when the sensors have identical sensing areas. Finally, we compute the fraction of the field of interest covered by at least k sensors.

A. Coverage Achieved by Random Deployment of a Single Sensor

Let us consider the simple case where a single sensor s_1 is randomly deployed in such a way that it monitors some part of the field of interest. The achieved coverage can be computed by considering the intersection of two sets in the plane. Let $\mathcal{A}_0, \mathcal{A}_1$ denote two sets in a plane with $\mathcal{A}_0$ being fixed, while $\mathcal{A}_1$ can move freely. $\mathcal{A}_0$ represents the field of interest, while $\mathcal{A}_1$ represents the sensing area of node s_1. The average size of the common area $\mathcal{A}_{01}$ between sets $\mathcal{A}_0, \mathcal{A}_1$, when $\mathcal{A}_1$ is randomly dropped in the plane, defines the area of $\mathcal{A}_0$, covered by $\mathcal{A}_1$. Normalizing $\mathcal{A}_{01}$ over $\mathcal{A}_0$ we obtain the fraction $fr(\mathcal{A}_0)$ of $\mathcal{A}_0$ covered by $\mathcal{A}_1$. In figure 1)(c), we show two sets $\mathcal{A}_l, \mathcal{A}_1$ and the common area between them.

To compute $fr(\mathcal{A}_0)$, we randomly select a point P of $\mathcal{A}_0$, and find the set of all positions of $\mathcal{A}_1$ that include P. Dividing the measure of all the positions of $\mathcal{A}_1$ that include P over the measure of all the positions of $\mathcal{A}_1$ such that $\mathcal{A}_0 \bigcap \mathcal{A}_1 \neq \emptyset$ yields the probability $p(P \in \mathcal{A}_1)$ that the randomly selected point P is covered by $\mathcal{A}_1$ [17], [18]. Integrating $p(P \in \mathcal{A}_1)$ over all $P \in \mathcal{A}_0$ and normalizing over the size of $\mathcal{A}_0$ yields $fr(\mathcal{A}_0)$. The following theorem holds only for convex sets, though it can be extended in the case of non-convex sets by appropriate computation of the kinematic measures [17], [18].

Theorem 1: Let $\mathcal{A}_0$ be a fixed convex set of area F_0 and perimeter L_0, and let $\mathcal{A}_1$ be a convex set of area F_1 and perimeter L_1, randomly dropped in the plane in such a way that it intersects with $\mathcal{A}_0$. The probability that a randomly selected point $P \in \mathcal{A}_0$ is covered by $\mathcal{A}_1$ is given by:

$$p(P \in \mathcal{A}_1) = \frac{2\pi F_1}{2\pi(F_0 + F_1) + L_0 L_1}. \tag{4}$$

Proof: The probability that P is covered by $\mathcal{A}_1$ is equal to the measure of the set of motions of $\mathcal{A}_1$ such that $P \in \mathcal{A}_1$ divided by the measure of the set of motions of $\mathcal{A}_1$ such that $\mathcal{A}_0 \bigcap \mathcal{A}_1 \neq \emptyset$. We now compute the two measures.

$$
\begin{aligned}
m(\mathcal{A}_1 : P \in \mathcal{A}_0 \bigcap \mathcal{A}_1) &\overset{(i)}{=} \int_{P \in \mathcal{A}_0 \bigcap \mathcal{A}_1} d\mathcal{A}_1 \\
&\overset{(ii)}{=} \int_{P \in \mathcal{A}_1} d\mathcal{A}_1 \\
&= \int_{P \in \mathcal{A}_1} dx \wedge dy \int_0^{2\pi} d\phi \\
&= 2\pi F_1, \tag{5}
\end{aligned}
$$

where in 5(i) we integrate $d\mathcal{A}_1$ over all motions of $\mathcal{A}_1$ such that $P \in \mathcal{A}_0 \bigcap \mathcal{A}_1$. Since by assumption $P \in \mathcal{A}_0$ and $\mathcal{A}_0$ is fixed, in 5(ii) we integrate $d\mathcal{A}_1$ over all motions of $\mathcal{A}_1$ such that $P \in \mathcal{A}_1$. The measure of all motions of $\mathcal{A}_1$ such that

$\mathcal{A}_0 \bigcap \mathcal{A}_1 \neq 0$ is:

$$
\begin{aligned}
m(\mathcal{A}_1 : \mathcal{A}_0 \bigcap \mathcal{A}_1 \neq \emptyset) &= \int_{\mathcal{A}_0 \bigcap \mathcal{A}_1 \neq \emptyset} d\mathcal{A}_1 \\
&= \int_{\mathcal{A}_0 \bigcap \mathcal{A}_1 \neq \emptyset} dx \wedge dy \wedge d\phi \\
&= 2\pi(F_0 + F_1) + L_0 L_1. \tag{6}
\end{aligned}
$$

Due to the length and complexity, the proof of (6) is omitted. Interested reader is referred to [17], [18] for details.

By combining (5) and (6) we can compute the probability $p(P \in \mathcal{A}_1)$ as:

$$
\begin{aligned}
p(P \in \mathcal{A}_1) &= \frac{m(\mathcal{A}_1 : P \in \mathcal{A}_0 \bigcap \mathcal{A}_1)}{m(\mathcal{A}_1 : \mathcal{A}_0 \bigcap \mathcal{A}_1 \neq \emptyset)} \\
&= \frac{2\pi F_1}{2\pi(F_0 + F_1) + L_0 L_1}. \tag{7}
\end{aligned}
$$

∎

Note that $p(P \in \mathcal{A}_1)$ is only dependent on the area and the perimeter of the convex sets that intersect and not on the shape of those sets.

Lemma 1: The fraction $fr(\mathcal{A}_0)$ of a fixed convex set $\mathcal{A}_0$ of area F_0 and perimeter L_0 that is covered by a convex set $\mathcal{A}_1$ of area F_1 and perimeter L_1, when $\mathcal{A}_1$ is randomly dropped in the plane in such a way that it intersects with $\mathcal{A}_0$ is given by:

$$fr(\mathcal{A}_0) = \frac{2\pi F_1}{2\pi(F_0 + F_1) + L_0 L_1}. \tag{8}$$

Proof: Equation (7) expresses the probability that a randomly selected point $P \in \mathcal{A}_0$ is covered by $\mathcal{A}_1$. Integrating (7) over all points $P \in \mathcal{A}_0$ provides the size F_{01} of the common area $\mathcal{A}_{01}$ between $\mathcal{A}_l$ and $\mathcal{A}_1$:

$$
\begin{aligned}
F_{01} &= \int_{P \in \mathcal{A}_0} p(P \in \mathcal{A}_1) dP \\
&= p(P \in \mathcal{A}_1) \int_{P \in \mathcal{A}_0} dP \\
&= p(P \in \mathcal{A}_1) F_0 \\
&= \frac{2\pi F_0 F_1}{2\pi(F_0 + F_1) + L_0 L_1}. \tag{9}
\end{aligned}
$$

Normalizing F_{01} by F_0 yields:

$$
\begin{aligned}
fr(\mathcal{A}_0) &= \frac{F_{01}}{F_0} \\
&= \frac{2\pi F_0 F_1}{2\pi(F_0 + F_1) + L_0 L_1} \frac{1}{F_0} \\
&= \frac{2\pi F_1}{2\pi(F_0 + F_1) + L_0 L_1} \\
&= p(P \in \mathcal{A}_1). \tag{10}
\end{aligned}
$$

∎

B. Coverage Achieved by Deployment of a Single Sensor According to an Arbitrary Distribution

In the case where the $\mathcal{A}_1$ is not randomly deployed in the plane, but it follows an arbitrary distribution $K(\mathcal{A}_0)$, the measures in (5), (6) are calculated as weighted functions of the probability density function $k(x, y, \phi)$ of $\mathcal{A}_1$.

$$m(\mathcal{A}_1 : \mathcal{A}_0 \bigcap \mathcal{A}_1 \neq \emptyset) = \int_Z k dx \wedge dy \wedge d\phi, \quad (11)$$

$$m(\mathcal{A}_1 : P \in \mathcal{A}_0 \bigcap \mathcal{A}_1) = \int_{P \in \mathcal{A}_1} k dx \wedge dy \wedge d\phi, (12)$$

where $Z = \mathcal{A}_0 \bigcap \mathcal{A}_1 \neq \emptyset$. Depending on the distribution $K(\mathcal{A}_0)$, the measures in (11), (12) may have a closed form. When $\mathcal{A}_\infty$ is deployed according to the distribution $K(\mathcal{A}_0)$, we can calculate the probability $p(P \in \mathcal{A}_1)$, by substituting the measures in (11), (12) into (7). The $p(P \in \mathcal{A}_1)$, is the basic building block for deriving expressions for coverage in the general case where N sensors are deployed, as we show in the following section.

C. Coverage in the Case of Multiple Sensors

In this section, we compute the probability $p(S = k)$ that a randomly selected point $P \in \mathcal{A}_0$ is covered by k sensors when N sensors are randomly deployed. Using $p(S = k)$, we compute the probability that P is covered by at least k sensors, as well as the fraction of $\mathcal{A}_0$ covered by at least k sensors.

Theorem 2: Let $\mathcal{A}_0$ be the field of interest of size F_0 and perimeter L_0, and let N sensors with sensing area $\mathcal{A}_i$ of size F_i and perimeter L_i be deployed over $\mathcal{A}_0$. The probability $p(S = k)$ that a randomly chosen point P of $\mathcal{A}_0$ is covered by *exactly k sensors* when $k \geq 1$ is given by:

$$p(S = k) = \frac{\sum_{i=1}^{\binom{N}{k}} \left(\prod_{j=1}^{k} (2\pi F_{T_{i,j}}) \prod_{z=1}^{N-k} \mathcal{J}(i, z) \right)}{\prod_{r=1}^{N} (2\pi(F_0 + F_r) + L_0 L_r)}, \quad (13)$$

where $\mathcal{J}(i, j) = (2\pi F_0 + L_0 L_{G_{i,z}})$, T is a matrix in which each row j is a k-permutation of $[1 \ldots N]$, and G is a matrix in which each row j contains the elements of $[1 \ldots N]$, that do not appear in the jth row of T.

Proof: In order to prove Theorem 2, we map the problem of coverage to the set intersection problem, as illustrated in our problem formulation in Section III-A. When a single sensor s_i is deployed, the probability that it covers a randomly selected point $P \in \mathcal{A}_0$ is given by Theorem 1. Hence, the probability $p(P \notin \mathcal{A}_i)$ can be computed as:

$$
\begin{aligned}
p(P \notin \mathcal{A}_i) &= 1 - p(P \in \mathcal{A}_i) \\
&= 1 - \frac{2\pi F_i}{2\pi(F_0 + F_i) + L_0 L_i} \\
&= \frac{2\pi F_0 + L_0 L_i}{2\pi(F_0 + F_i) + L_0 L_i}. \quad (14)
\end{aligned}
$$

Given that fact that the N sensors are *independently* deployed in the plane so that they cover some part of $\mathcal{A}_0$, the probability $p(S = k)$ that a randomly selected point $P \in \mathcal{A}_0$ is covered by

exactly k sensors is equal to the probability that P is covered by exactly k specific sets. Let T denote a $k \times \binom{N}{k}$ matrix where each row j is a k-permutation of the vector $[1 \ldots N]$, and let G denote a $(N-k+1) \times \binom{N}{k}$ matrix where each row j contains the elements of $[1 \ldots N]$, that do not appear in the jth row of T. Consider for example, $T(1) = [1 \ldots k]$ and $G(1) = [k + 1 \ldots N]$. The probability $p(T(1))$ that P is covered by exactly the sets with indexes in the first row of T is given by:

$$
\begin{aligned}
p(T(1)) &\overset{(i)}{=} p(P \in \mathcal{A}_1, \ldots, P \notin \mathcal{A}_{k+1}, \ldots, P \notin \mathcal{A}_N) \\
&\overset{(ii)}{=} p(P \in \mathcal{A}_1), \ldots, (P \in \mathcal{A}_k) \\
&\quad p(P \notin \mathcal{A}_{k+1}), \ldots, p(P \notin \mathcal{A}_N) \\
&\overset{(iii)}{=} \frac{2\pi F_1}{2\pi(F_0 + F_1) + L_0 L_1} \\
&\cdots \frac{2\pi F_k}{2\pi(F_0 + F_k) + L_0 L_k} \\
&\quad \frac{2\pi F_0 + L_0 L_{k+1}}{2\pi(F_0 + F_{k+1}) + L_0 L_{k+1}} \\
&\cdots \frac{2\pi F_0 + L_0 L_N}{2\pi(F_0 + F_N) + L_0 L_N} \\
&= \frac{\prod_{j=1}^{k}(2\pi F_j) \prod_{z=k+1}^{N}(2\pi F_0 + L_0 L_z)}{\prod_{r=1}^{N}(2\pi(F_0 + F_r) + L_0 L_r)} \\
&= \frac{\prod_{j=1}^{k}(2\pi F_{T_{1,j}}) \prod_{z=1}^{N-k}(2\pi F_0 + L_0 L_{G_{1,z}})}{\prod_{r=1}^{N}(2\pi(F_0 + F_r) + L_0 L_r)}.
\end{aligned}
$$

$$(15)$$

In (i), we show which k sets include point P. Due to the independence in the set deployment, in (ii), the intersection of the events in (i) becomes a product of the individual events. In (iii), we substitute the individual probabilities from (7), (14). In the general case, the probability that the sets with indexes of the ith row of T cover point P is given by:

$$p(T(i)) = \frac{\prod_{j=1}^{k} (2\pi F_{T_{i,j}}) \prod_{z=1}^{N-k} \mathcal{J}(i, z)}{\prod_{r=1}^{N} (2\pi(F_0 + F_r) + L_0 L_r)}. \quad (16)$$

Since we are not interested in a specific set permutation to cover point P, the probability that $p(S = k)$ is a summation of $p(T(i))$ for all possible k-permutations. Summing $p(T(i))$ over all i yields (13):

$$
\begin{aligned}
p(S = k) &= \sum_{i=1}^{\binom{N}{k}} p(T(i)) \\
&= \sum_{i=1}^{\binom{N}{k}} \left(\frac{\prod_{j=1}^{k} (2\pi F_{T_{i,j}}) \prod_{z=1}^{N-k} \mathcal{J}(i, z)}{\prod_{r=1}^{N} (2\pi(F_0 + F_r) + L_0 L_r)} \right)
\end{aligned}
$$

$\blacksquare$

According to Lemma 1, (13) also expresses the fraction of $\mathcal{A}_0$ that is covered by exactly k sensors. Equation (13) is valid for $k \geq 1$. The fraction of the $\mathcal{A}_0$ that is not covered by any sensor, is given by the following corollary.
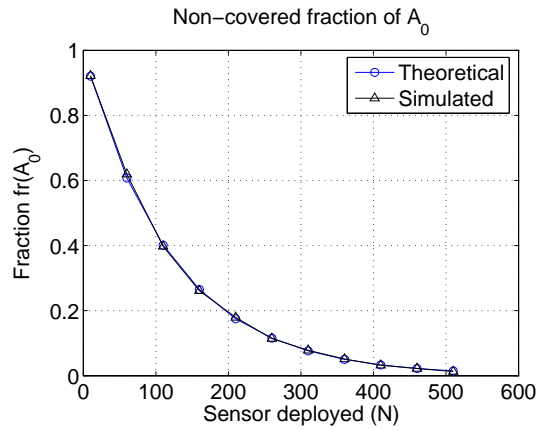
Fig. 2. Fraction $fr(\mathcal{A}_0)$ of $\mathcal{A}_0$, that remains non-covered as a function of the number of sensors N that are deployed to monitor the field of interest.

Corollary 1: The fraction of $\mathcal{A}_0$ that is not covered by any sensor when N sensors are randomly deployed is given by,

$$p(S = 0) = \prod_{i=1}^{N} \frac{2\pi F_0 + L_0 L_i}{2\pi(F_0 + F_i) + L_0 L_i}. \qquad (17)$$

Proof: Given that fact that the N sensors are *independently* deployed in the plane so that they cover some part of $\mathcal{A}_0$, the probability $p(S = 0)$ that none of the $\mathcal{A}_i$, $i = 1 \ldots N$ covers point P is:

$$
\begin{aligned}
p(S = 0) &= p(P \notin \mathcal{A}_1, \ldots, P \notin \mathcal{A}_N) \\
&\overset{(i)}{=} \prod_{i=1}^{N} p(P \notin \mathcal{A}_i) \\
&\overset{(ii)}{=} \prod_{i=1}^{N} \left(\frac{2\pi F_0 + L_0 L_i}{2\pi(F_0 + F_i) + L_0 L_i} \right). \qquad (18)
\end{aligned}
$$

Equality in (i) holds due to the independence in the deployment of the sensors s_i. In (ii), we substitute $p(P \notin \mathcal{A}_i)$ from (14). ∎

In the case where the sensors have identical sensing area, that is, $F_i = F$ and $L_i = L$ then the following corollary holds.

Corollary 2: Let $F_i = F$ and $L_i = L$. The probability that a randomly selected point of $\mathcal{A}_0$ is covered by exactly k sensors is given by

$$p(S = k) = \frac{\binom{N}{k}(2\pi F)^k(2\pi F_0 + L_0 L)^{N-k}}{(2\pi(F_0 + F) + L_0 L)^N}. \qquad (19)$$

Proof: Corollary 2 holds by substituting $F_i = F$ and $L_i = L$, into (13). ∎

Once we have computed the probability for a randomly selected point P of $\mathcal{A}_0$ to be covered by exactly k sensors, we can also compute the probability that a randomly selected point P is covered by *at least* k sensors.

Theorem 3: Let $\mathcal{A}_0$ be the field of interest of size F_0 and perimeter L_0, and let N sensors with sensing area $\mathcal{A}_i$ of size F_i and perimeter L_i be deployed over $\mathcal{A}_0$. The probability that a randomly selected point of $\mathcal{A}_0$ is covered by *at least* k sensors is given by:

$$p(S \geq k) = 1 - \sum_{h=1}^{k-1} p(S = h) \qquad (20)$$

where

$$p(S = h) = \frac{\sum_{i=1}^{\binom{N}{h}} \left(\prod_{j=1}^{h}(2\pi F_{T_{i,j}}) \prod_{z=1}^{N-h} \mathcal{J}(i, z) \right)}{\prod_{r=1}^{N}(2\pi(F_0 + F_r) + L_0 L_r)}.$$

Proof: Theorem 3, holds by observing:

$$p(S \geq k) = 1 - p(S < k) = 1 - \sum_{h=1}^{k-1} p(S = h), \qquad (21)$$

and substituting (13) into (21). ∎

V. VALIDATION OF THE THEORETICAL RESULTS

In this section, we validate our theoretical results derived in Section IV via simulation. We perform experiments for both homogeneous and heterogeneous sensor networks and show that the theoretical formulas match the simulations. We also provide an example for analytically computing the number of sensors that need to be deployed in order to achieve the desired degree of coverage.

A. Coverage in Homogeneous Sensor Networks

In our first experiment, we randomly deployed a variable number of sensors with identical sensing area in a disk of radius $R = 100$m. All sensors had a circular sensing area of radius $r = 10$m. We repeated the experiments 100 times and averaged the results. We first compute the fraction $fr(\mathcal{A}_0)$ of $\mathcal{A}_0$, that remains non-covered as a function of the number of sensors N that are deployed to monitor the field of interest.
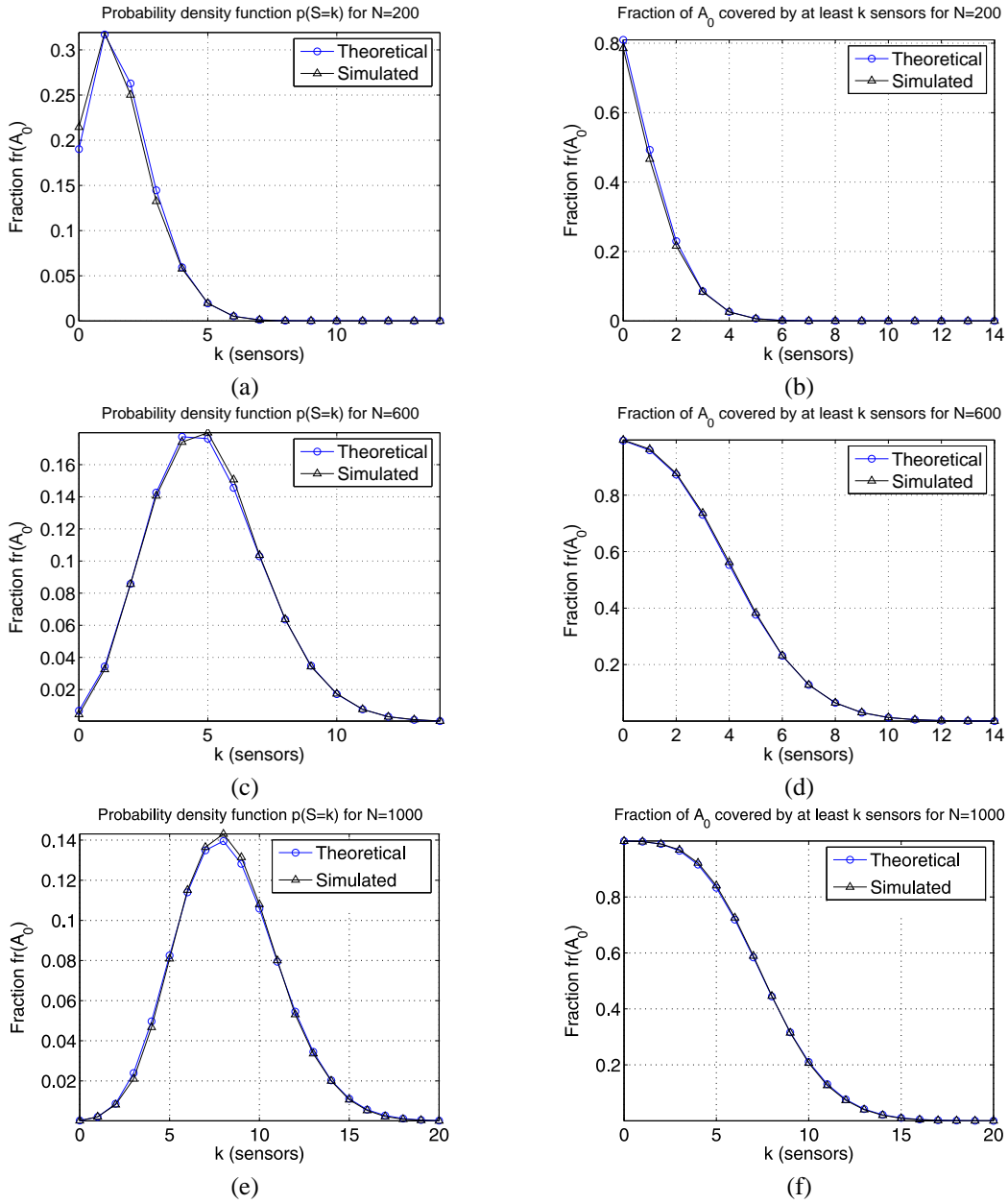
Fig. 3. (a) The pdf of the fraction $fr(\mathcal{A}_0)$ covered by exactly k sensors when $N = 300$ sensors with identical sensing area are randomly deployed. (b) The fraction $fr(\mathcal{A}_0)$ covered by at least k sensors when $N = 300$ sensors with identical sensing area are randomly deployed. (c) The pdf of the fraction $fr(\mathcal{A}_0)$ covered by exactly k sensors when $N = 500$ sensors with identical sensing area are randomly deployed. (d) The fraction $fr(\mathcal{A}_0)$ covered by at least k sensors when $N = 500$ sensors with identical sensing area are randomly deployed. (a) The pdf of the fraction $fr(\mathcal{A}_0)$ covered by exactly k sensors when $N = 1000$ sensors with identical sensing area are randomly deployed. (f) The fraction $fr(\mathcal{A}_0)$ covered by at least k sensors when $N = 1000$ sensors with identical sensing area are randomly deployed.

The theoretical formula that computes $fr(\mathcal{A}_0)$ is obtained from Corollary 1 and is equal to:

$$fr(\mathcal{A}_0) = p(S = 0) = \left(\frac{2\pi F_0 + L_0 L}{2\pi(F_0 + F) + L_0 L} \right)^N, \quad (22)$$

where $F_0 = \pi R^2, L_0 = 2\pi R, F = \pi r^2, L = 2\pi r$. In figure 2(a), we show the fraction $fr(\mathcal{A}_0)$ of $\mathcal{A}_0$, that remains non-covered as a function of the number of sensors N that are deployed to monitor the field of interest. We observe that

the theoretical formula in (22) conforms with the simulation results. Since our method does not suffer from the border effect problem, (22) is accurate despite the bounded size of the field of interest.

In figure 3(a), we show the pdf of the fraction $fr(\mathcal{A}_0)$ covered by exactly k sensors when $N = 200$ sensors with identical sensing area are randomly deployed. The equivalent sensor density is equal to $\rho = 0.0063$ sensors/m^2. The same graphs for $N = 600, N = 1000$ (densities $\rho = 0.019$

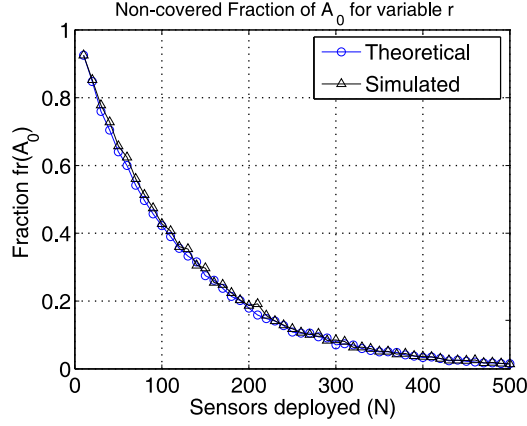Fig. 4. Fraction $fr(\mathcal{A}_0)$ of $\mathcal{A}_0$, that remains non-covered as a function of the number of sensors N that are deployed to monitor the field of interest.

sensors/m^2, $\rho = 0.032$ sensors/m^2) are provided in figures 3(c) and 3(e), respectively. The pdf of $fr(\mathcal{A}_0)$ is equal to the probability that a randomly selected point P is covered by exactly k sensors. Our analytical derivation in Section IV, yields:

$$
\begin{aligned}
fr(\mathcal{A}_0) &= p(S = k) \\
&= \frac{\binom{N}{k}(2\pi F)^k (2\pi F_0 + L_0 L)^{N-k}}{(2\pi(F_0 + F) + L_0 L)^N}. \quad (23)
\end{aligned}
$$

In figure 3(b), we show the fraction of $\mathcal{A}_0$ covered by *at least* k sensors when $N = 200$. The same graphs for $N = 600, N = 1000$ are provided in figure 3(d) and 3(f), respectively. For both values of N we observe that our theoretic formulas conform with the simulation results. For all graphs in figure 2, 3 we show the theoretical result according to our expressions, and the simulation values.

B. Coverage in Heterogeneous Sensor Networks

In our second experiment, we considered a hierarchical (heterogeneous) sensor network, where two types of sensors are deployed. Type A has a sensing area of disk shape with a sensing range $r_A = 10m$, while type B has a sensing area of disk shape with a sensing range of $r_B = 15m$. We randomly deployed an equal number $N_A = N_B = \frac{N}{2}$ of sensors of each type over a circular field of interest of size $F_0 = \pi R^2$ where $R = 100m$. In figure 4, we show the fraction $fr(\mathcal{A}_0)$ of $\mathcal{A}_0$, that remains non-covered as a function of the number of sensors N that are deployed to monitor the field of interest. The theoretical formula that compute that is equal to:

$$
\begin{aligned}
fr(\mathcal{A}_0) &= p(S = 0) \\
&= \prod_{i=1}^{N} \frac{2\pi F_0 + L_0 L_i}{2\pi(F_0 + F_i) + L_0 L_i}, \quad (24)
\end{aligned}
$$

where $F_0 = \pi R^2, L_0 = 2\pi R, F_i = \pi r_i^2, L = 2\pi r_i$.

We observe that the simulation results verify the validity of our theoretical expression. In figure 5(a), we show the pdf of the fraction $fr(\mathcal{A}_0)$ covered by exactly k sensors when $N = 300$ sensors are randomly deployed. The equivalent

sensor density is equal to $\rho = 0.0095$ sensors/m^2. The same graph for $N = 500, N = 1000$ (densities $\rho = 0.019$ sensors/m^2, $\rho = 0.032$ sensors/m^2) are provided in figures 5(c) and 5(e), respectively. The $fr(\mathcal{A}_0)$ covered by exactly K sensors is equal to the pdf $p(S = k)$ of the probability that a randomly selected point P is covered by exactly k sensors. Our analytical derivation in Section IV, yields:

$$
fr(\mathcal{A}_0) = p(S = k)
$$

For $k = 0$:

$$
fr(\mathcal{A}_0) = \prod_{i=1}^{N} \left(\frac{2\pi F_0 + L_0 L_i}{2\pi(F_0 + F_i) + L_0 L_i} \right),
$$

while for $k \geq 1$:

$$
fr(\mathcal{A}_0) = \frac{\sum_{i=1}^{\binom{N}{k}} \left(\prod_{j=1}^{k} (2\pi F_{T_{i,j}}) \prod_{z=1}^{N-k} \mathcal{J}(i,z) \right)}{\prod_{r=1}^{N} (2\pi(F_0 + F_r) + L_0 L_r)}.
$$

In figure 5(b), we show the fraction of $\mathcal{A}_0$ covered by *at least* k sensors when $N = 300$. The same graphs for $N = 500, N = 1000$ are provided in figures 5(d), and 5(f), respectively. We again verify that our theoretical formula agrees with the simulation results.

In the case of heterogeneous sensor networks where each sensor has a different sensing area, the formula in (25) has an exponentially increasing computational cost, since an exponentially increasing summation of terms must be computed in order to derive the exact coverage achieved. Such a computation may not be feasible for large networks. In such a case, an approximation can be used for our formulas by employing the expressions derived for a homogeneous sensor network and substituting the size F and perimeter L of the sensing area of the sensors with the expected size $E[F]$ and expected perimeter $E[L]$. The theoretical approximation for such a case is:

$$
\begin{aligned}
fr(\mathcal{A}_0) &= p(S = k) \\
&= \frac{\binom{N}{k}(2\pi E[F])^k (2\pi F_0 + L_0 E[L])^{N-k}}{(2\pi(F_0 + E[F]) + L_0 E[L])^N}. \quad (25)
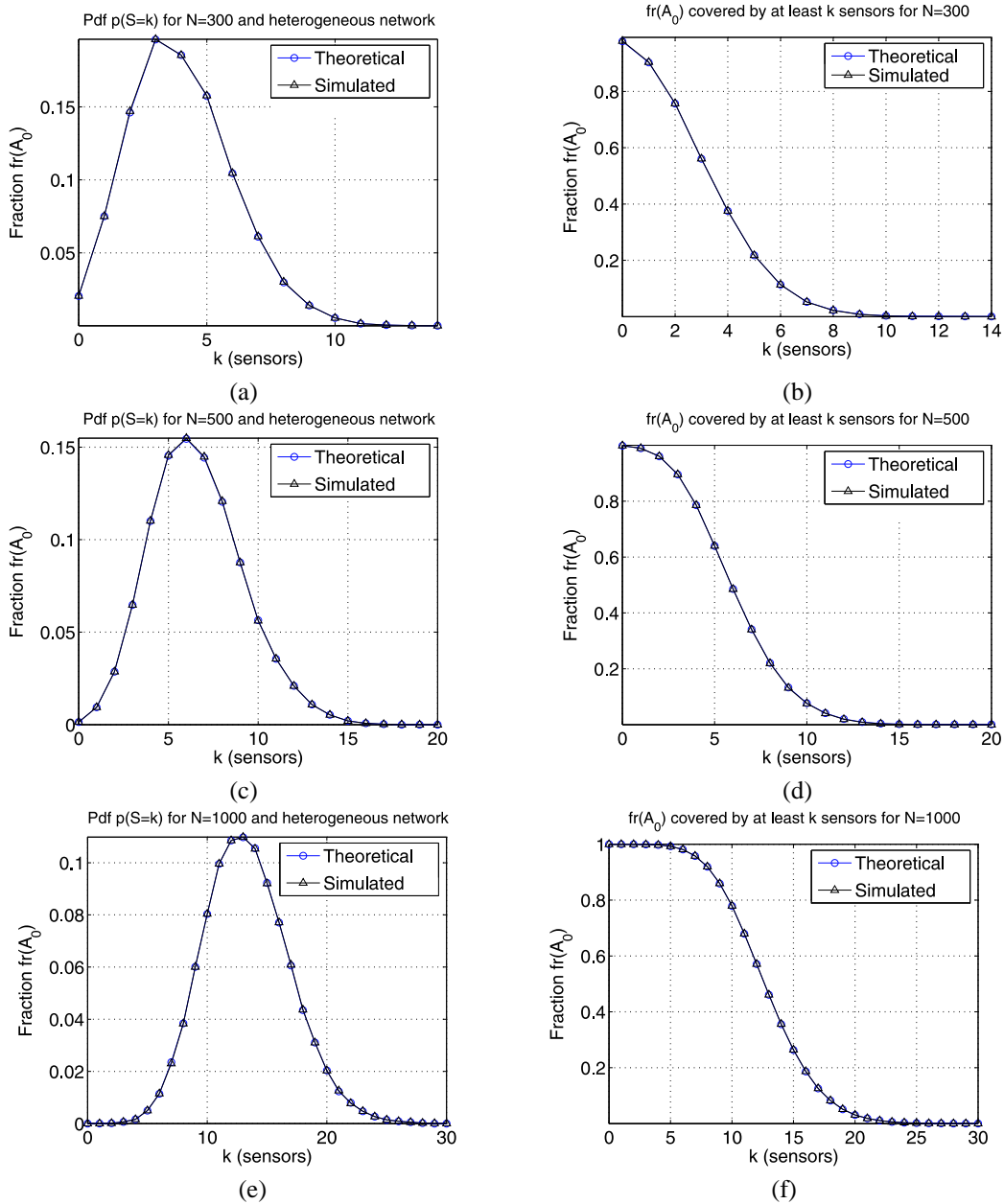\end{aligned}
$$

Fig. 5. Heterogeneous sensor network, with the field of interest being a disk of radius $R = 100m$. An equal number of two types of sensors are deployed; Type A has a sensing area of a disk shape with radius $r_A = 10m$, while type B has a sensing area of a disk shape with $r_B = 15m$. (a) The pdf of the fraction $fr(\mathcal{A}_0)$ covered by exactly k sensors when $N = 300$ sensors. (b) The fraction $fr(\mathcal{A}_0)$ covered by at least k sensors when $N = 300$ sensors. (c) The pdf of the fraction $fr(\mathcal{A}_0)$ covered by exactly k sensors when $N = 500$ sensors. (d) The fraction $fr(\mathcal{A}_0)$ covered by at least k sensors when $N = 500$ sensors. (e) The pdf of the fraction $fr(\mathcal{A}_0)$ covered by exactly k sensors when $N = 1000$ sensors. (f) The fraction $fr(\mathcal{A}_0)$ covered by at least k sensors when $N = 1000$ sensors.

In figure 6(a) we show the pdf obtained via simulation for our heterogeneous sensor network experiment, for $N = 500$ sensors, the theoretical values based on the exact formula in (25), and the approximation in (25). In figure 6(b), we show the fraction of $\mathcal{A}_0$ covered by at least k sensors. We observe that for the case of heterogeneous sensor networks where each sensor has a different sensing area, (25) provides a good approximation of the coverage achieved, without incurring the computational cost of (25).

C. An Example of Computing the Coverage in a Sample Network

In this section, we provide an example of applying our results to a sample sensor network. Consider an FoI of size $F_0 = 10^6 m^2$ and perimeter $L_0 = 4,000m$ where sensors of identical sensing area $F = 100\pi$ and perimeter $L = 20\pi$ are randomly deployed. We want to compute the number of sensors needed in order for a randomly selected point of the FoI to be covered by at least one sensor with a probability
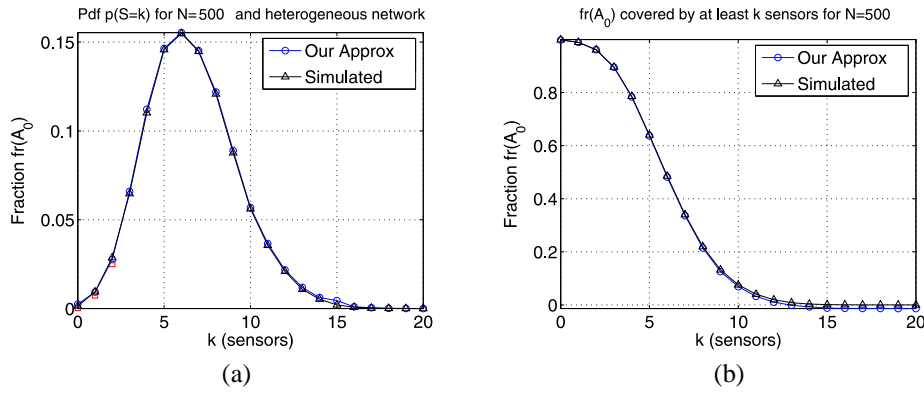
Fig. 6. Heterogeneous sensor network, with the field of interest being a disk of radius $R = 100m$. An equal number of two types of sensors are deployed; Type A has a sensing area of a disk shape with radius $r_A = 10m$, while type B has a sensing area of a disk shape with $r_B = 15m$. (a) The pdf of the fraction $fr(\mathcal{A}_0)$ covered by exactly k sensors when $N = 500$ sensors. (b) The fraction $fr(\mathcal{A}_0)$ covered by at least k sensors when $N = 500$ sensors.

$p_C = 95\%$. Or alternatively, the number of sensors N needed, so that a fraction $p_C = 0.95$ of the field of interest is covered by at least one sensor. Corollary 1 yields:

$$
\begin{aligned}
p(S \geq 1) &= 1 - p(S = 0) \\
&= 1 - \prod_{i=1}^{N} \left(\frac{2\pi F_0 + L_0 L}{2\pi (F_0 + F) + L_0 L} \right) \\
&= 1 - \left(\frac{2\pi F_0 + L_0 L}{2\pi (F_0 + F) + L_0 L} \right)^N.
\end{aligned}
$$

We want to the probability of 1-coverage to be at least $p(S \geq 1) \geq p$. Hence,

$$
\begin{aligned}
P(S \geq 1) &= 1 - \left(\frac{2\pi F_0 + L_0 L}{2\pi (F_0 + F) + L_0 L} \right)^N \geq p_C \Rightarrow \\
N &\geq \frac{\log (1 - p_C)}{\log \left(\frac{2\pi F_0 + L_0 L}{2\pi (F_0 + F) + L_0 L} \right)}.
\end{aligned}
$$

Substituting the values for p_C, F_0, L_0, F, L yields $N \geq 9,728$ sensors.

VI. CONCLUSION

We studied the problem of stochastic coverage in planar heterogeneous sensor networks. We formulated the coverage problem as a set intersection problem and used results from Integral Geometry to obtain analytical expressions for the coverage achieved by the deployment of N sensors. Our formulation generalizes to a heterogeneous sensing model where each sensor has a different sensing area, while it does not suffer from the border effects problem. Furthermore, our approach applies to sensor deployment according to any distribution. To verify our results, we performed extensive simulation and showed that the simulation conforms with our theoretic formulas.

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