Link Delay Estimation via Expander Graphs

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Abstract

In network tomography, we seek to infer the status of parameters (such as delay) for links inside a network through *end-to-end* probing between (external) boundary nodes along predetermined routes. In this work, we apply concepts from compressed sensing for network topologies that are expanders, to the delay estimation problem. We first show that a relative majority of network topologies are *not* expanders for the existing error bounds. Motivated by this, we relax this bound leading to evidence that for 30% more networks, the link delays can be estimated. We provide simulation performance analysis of delay estimation based on l_1 minimization, showing that accurate estimation is feasible for an increasing proportion of networks.

Index Terms

Network Tomography, Delay Estimation, Compressed Sensing, Expander Graphs, l_1 minimization

I. Introduction

Monitoring of link properties (delay, loss rates, etc.) in networks continues to be an integral requirement within any network management framework as part of monitoring its utilization and performance. The need for accurate and fast monitoring schemes has escalated in recent years due to the increasing popularity of new resource-consuming services (such as video-conferencing, Voice over IP, and online games) that require quality-of-service (QoS) guarantees [2]. The primary objective of this paper is to demonstrate how compressed sensing ideas may be applied to derive a fast delay monitoring algorithm that outperforms other schemes.

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A preliminary version of our results appears in [1]. This submission is distinct due to the fuller discussion of $(k > 1)$-identifiability, a thorough set of simulation experiments, and including the proofs of theorems.

The term *network tomography* was used in [3] to encompass a class of approaches that seek to infer the internal link status from end-to-end measurements [4], [5], [6]. A useful classification of network tomography methods for our purposes is as follows [7]:

- **Cooperative Internal Nodes:** This method assumes that internal nodes on probe routes respond to *control* packets. For example, active tools such as a ping or a trace route, measure and report attributes of the round-trip path from a sender to the internal node based on probe packets [8]. Beside complexity, the challenges of such methods arise from the fact that service providers do not own the entire network that is being probed and hence do not have access to the desired internal nodes for appropriate configuration [9], [10].
- **End-to-End:** In networks with a defined *boundary*, it is assumed that access is available to (all) nodes at the edge (but not to any in the interior). A boundary node sends probes to all (or a subset of) other boundary nodes to measure packet attributes on the path between network end points. These edge-based methods do not require exchanging control messages with any interior nodes. The primary challenge confronting such end-to-end probe-based link status estimation is that of identifiability, as discussed below [11], [12], [2].

As the networks evolve toward more decentralized, uncooperative, and heterogeneous administrative (sub)domains, the availability of cooperative interior nodes is increasingly limited. Hence, end-to-end network diagnostic tools attract increasing attention. In end-to-end network tomography, probes are sent between boundary nodes on *predetermined* routes; typically, these are usually the shortest paths between the nodes based on existing routing protocols.

For parameters such as delay, an additive linear model adequately represents the relationship between a measured path and an individual link delay [13], [14], i.e.,

$$\mathbf{y} = \mathbf{R}\mathbf{x}, \tag{1}$$

where $\mathbf{x}$ is the $n \times 1$ (unknown) vector of the individual link mean delay. The $r \times n$ *binary* matrix $\mathbf{R}$ is the routing matrix for the network graph corresponding to the paths chosen for the probes (note: each row of the matrix correspond to a path), and $\mathbf{y} \in \mathbb{R}^r$ is the measured r-vector of end-to-end path delays. Although the focus of this paper is link delay, our approach readily applies to any other link attributes (such as $\log$ of packet loss rate), allowing such a linear relationship with end-to-end measurements.

Link delay estimators based on Eq. (1) can be classified as follows:

1) **Deterministic**: The delays are considered unknown but constant. Because the link delay is typically time varying, such approaches are suitable for periods of local "stationarity" where such an assumption is valid.
2) **Stochastic**: The delay vector $\mathbf{x}$ is specified by a suitable a-priori parametric probability distribution; the method then estimates the unknown parameters of the model. For example, [15], [13], [14], [16] assumed that link delay follows a Gaussian or an exponential distribution.

Both modeling approaches have their pro's and con's. Stochastic models are usually more computationally intensive than deterministic ones [17] as they suffer from overmodeling (too many parameters for the data). Moreover, in many scenarios, one is typically interested in only the *few* links that are congested (i.e., suffer excessive link delay). Deterministic models are better suited to exploit this (side) information; our method falls within this class.

In Eq. (1), usually, the number of observations r is much less than the number of variables n (i.e., $r \ll n$) because the number of accessible boundary nodes is much smaller than the number of links inside the network. Thus, the number of variables in Eq. (1) to be estimated is much larger than the number of equations [16], leading to the generic nonuniqueness of solutions to Eq. (1), i.e., the inability to uniquely determine link delay [15] from end-to-end measurements. However, the problem of identifying only the (few) links with large delays (a.k.a congested links[1]) suggests the possibility of improved mechanisms to solve the under-determined system in Eq. (1), provided that the *sparsity* of the desired solution can be exploited. In other words, we are interested in solutions $\mathbf{x}$ with only a few - upto k large entries. If the other entries are small, we refer to such vector as *nearly k-sparse*, and if they are exactly zero we call it *exactly k-sparse*. Clearly, if vector $\mathbf{x}$ is exactly k-sparse, it is also nearly k-sparse. For the sake of simplicity, we use the terms *nearly k-sparse* and *k-sparse* interchangeably in the sequel. A network is called k-identifiable if for every *exactly k-sparse delay vector* $\mathbf{x}$, Eq. (1) is uniquely solvable.

Compressed sensing has been proposed recently for network tomography [18], [18], [19], [20] as part of methods that vary significantly in their underlying assumptions and utility for practical networking scenarios. Authors in [18] used compressed sensing to estimate link delays

[1]A congested link is one with a significantly elevated delay compared to the rest of the links in the network.

of the unobserved links on an end-to-end path when measured data is available on a subset of links. Xu et. al. [19] applied compressed sensing by performing a standard random walk over a *sufficiently* connected graph to take measurements. However, this is at variance with typical network scenarios where the measurement matrix (i.e., routing matrix) is already given. Besides, most networks are not sufficiently connected [21], [22]. In this work we assume that the routing path between any pair of boundary nodes is predetermined–usually by shortest path algorithm–without any constraint on the underlying network topology.

This work applies the concepts of *expander graphs* to the network tomography problem along with compressive sensing based link delay estimation [23], [24], [25], [26]. This is achieved by fundamentally relating the network routing matrix to a bipartite graph. If the bipartite graph is an *expander graph*, then one can use l_1 minimization to solve Eq. (1), that has polynomial complexity in n, independent of k [27]. We derive the proposed delay estimation algorithm for network topologies that are expanders for the case $k = 1$ initially largely for illustrative purposes. The remainder of the paper then focusses on the general $k > 1$ case.

A. Contributions and Organization

Our specific contributions are as follows: we first establish a novel connection between network delay tomography and binary compressed sensing via the notion of expander graphs. Next, for 1-identifiable networks, we relax the existing result for expansion from $\epsilon \leq 1/6$ to $\epsilon \leq 1/4$ (Lemma 1). Further, we extend our result for expander graphs to include networks that are union of sub-graphs which are themselves expanders in Theorem 2. We then provide simulation results to show that in general ($k > 1$), a large proportion of networks (more than 60%) do *not* satisfy the conditions for being an expander. Hence, we derive new results that broaden the set of potential expanders at the cost of accepting a bigger error margin in reconstruction for the general k case (Theorem 3). We derive estimation error bounds for l_1 minimization link delays that are validated by simulation results. Our simulation evidence shows that the proposed delay estimator achieves predicted accuracy for a larger fraction of networks, compared to the state-of-the-art in the literature [28], [29].

The rest of the paper is organized as follows: Section II relates the routing matrix of a network to bipartite graphs. Section III establishes a connection between link delay estimation and binary compressed sensing and identifies conditions on the network routing matrix under which a given network is k-identifiable. We evaluate our findings using simulations in Section IV. The paper concludes with reflections on possible future works in Section V. In the Appendix we provide proof of the theorems.

Notations: We use bold capitals (e.g. $\mathbf{R}$) to represent matrices and bold lowercase symbols (e.g. $\mathbf{x}$) for vectors. The i-th entry of a vector $\mathbf{x}$ is denoted by x_i. For the matrix $\mathbf{R}$, $\mathcal{N}(\mathbf{R})$ denotes its null space, and superscript ^{t} denotes its transpose. A set is denoted by a normal capitals (e.g. V) and a set of sets is presented by calligraphic capitalized symbol, e.g. $\mathcal{R}$ which is the set of all end-to-end paths in the network[2]. $|\mathcal{R}|$ is the cardinality (number of elements) in the set. An empty set is denoted by $\emptyset$. $deg(v)$ indicates degree of the node v in a graph, defined as number of nodes it is connected to.

For any set $S \subset \{1, 2, 3, ...n\}$, S^c represents the complement. Also, for any vector $\mathbf{x} \in \mathbb{R}^n$, vector $\mathbf{x}_S \in \mathbb{R}^n$ has entries defined as follows:

$$(x_S)_i = \begin{cases} x_i & \text{if } i \in S \\ 0 & \text{o.w.} \end{cases}. \tag{2}$$

If $\mathbf{x} \in \mathbb{R}^n$, the l_p-norm of $\mathbf{x}$ is defined as follows:

$$\| \mathbf{x} \|_p = \left(\sum_{i=1}^{n} x_i^p \right)^{\frac{1}{p}}. \tag{3}$$

II. Routing Matrix and Bipartite Graph

As is customary, a network consisting of bidirectional links connecting transmitters, switches, and receivers can be modeled as an undirected graph $N(V, E)$, where V (E) is the set of vertices (edges). Throughout this manuscript, boundary nodes are depicted as solid circles, while intermediate nodes are presented using dashed circles. We use network depicted in Figure 1 to illustrate the subsequent definitions.

In this section, we show that the routing matrix of any network can be represented as a *bi-adjacency matrix* of a suitably defined *bipartite* graph. This will help connect the problem of network identifiability with *expander graphs*, a special subset of bipartite graphs.

Definition 1. *A bipartite graph is one whose vertices can be divided into two disjoint sets, X and Y, so that every edge connects a vertex in X to one in Y [30].*

[2]Each path is itself set of nodes and edges of the network.

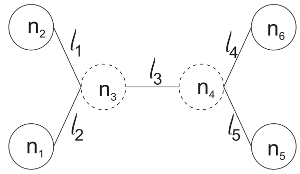


Fig. 1. A network with 4 boundary nodes, 2 intermediate nodes and 5 links

A bipartite graph is usually represented as a triple $G(X, Y, H)$, where $H \subset X \times Y$ is a set with paired elements from X and Y. The vertex sets X and Y are the left and right sides of the graph, respectively. A bipartite graph $G(X, Y, H)$ can be represented by its *bi-adjacency* matrix $\mathbf{A} = [a_{ij}]$, where $a_{ij} = 1$ if node $i \in Y$ is connected to node $j \in X$, and is zero otherwise, i.e.,

$$\begin{aligned} a_{ij} &= \begin{cases} 1 & (j,i) \in H \\ 0 & (j,i) \notin H \end{cases}, \\ \mathbf{A} &= [a_{ij}]. \end{aligned} \tag{4}$$

Note that in the definition of the bi-adjacency matrix $\mathbf{A}$, rows of $\mathbf{A}$ correspond to Y, which is the right-hand side of the graph; columns of $\mathbf{A}$ correspond to X, which is the left-hand side of the graph. This convention is used throughout the paper.

Assume that a given network $N(V, E)$ has a total of n links (i.e., $n = |E|$), and $\mathcal{R}$ is the (given) set of paths between the boundary nodes of the network and $r = |\mathcal{R}|$. Let $\mathbf{R}_{r\times n}$ denote the routing matrix, where there exists an isomorphism between the set $\mathcal{R}$ and the corresponding routing matrix $\mathbf{R}$. For example, for the 1-identifiable network[3] in Figure 1, suppose the following routing matrix is given:

$$\mathbf{R} = \begin{array}{c} \\ P_1 : n_2 \rightsquigarrow n_6 \\ P_2 : n_1 \rightsquigarrow n_5 \\ P_3 : n_1 \rightsquigarrow n_2 \\ P_4 : n_5 \rightsquigarrow n_6 \end{array} \begin{array}{c} \begin{array}{ccccc} l_1 & l_2 & l_3 & l_4 & l_5 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix} \end{array}, \tag{5}$$

[3]A network with no degree-two nodes is known to be 1-identifiable.

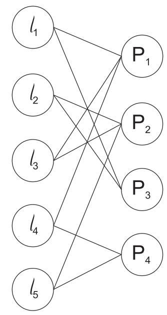


Fig. 2. Bipartite graph corresponding to given routing matrix in Eq. (5)

which is equivalent to the following set of paths $\mathcal{R}$:

$$\mathcal{R} = \{l_1 l_3 l_4, l_2 l_3 l_5, l_1 l_2, l_4 l_5\}. \tag{6}$$

$\mathbf{R}_{r\times n}$ can be viewed as a bi-adjacency matrix of a bipartite graph $G(X, Y, H)$, where $X = E$ (set of links in the network) and $Y = \mathcal{R}$ (set of given paths in the network). There exists a connection between a node in X and a node in Y if a path in $\mathcal{R}$ includes the corresponding link in E. Figure 2 presents the bipartite graph for the network in Figure 1 with the routing matrix $\mathbf{R}$ in Eq. (5).

Note that the above routing matrix, or its equivalent set of paths, is not a complete set of routes for the network in Figure 1 (e.g., it does not include the path from n_1 to n_6, which is $l_2 l_3 l_4$). However, it is a fundamental premise in network tomography that the routing matrix is already chosen and may not be changed. Hence, we initially seek to investigate the following question: Assuming that the routing matrix is given, when is it possible to identify or estimate link delays?

III. Expander Graphs and Network Identifiability

In recent years, a new approach–*Compressed Sensing*–for estimating an n-dimensional (signal) vector $\mathbf{x}$ from a lower-dimensional representation has attracted much attention [26], [31], [25]. For any signal $\mathbf{x} \in \mathbb{R}^n$, the reduced dimension representation is equal to $\mathbf{y} = \mathbf{A}\mathbf{x}$, where $m \times n$ matrix $\mathbf{A}$ ($m << n$) is referred to as the *measurement matrix*. The main challenge in traditional compressed sensing is to construct $\mathbf{A}$ with the following desirable (and conflicting) properties: (a) achieve maximum possible compression (m/n small) and yet allow (b) an accurate reconstruction of $\mathbf{x}$ from $\mathbf{y}$ when $\mathbf{x}$ is known to be sparse using (c) a fast decoding algorithm

[32], [33], [34], [35]. For example, when $\mathbf{A}$ is a binary matrix, $m = O(k \, log\frac{n}{k})$ suffices when $\mathbf{x}$ is nearly k-sparse.

As discussed above, the routing matrix of a network is the measurement matrix for delay tomography application, and in most scenarios it is predetermined. The main issue, therefore, is to determine whether it is an *appropriate* measurement matrix for compressed sensing, i.e., if it satisfies objective (b) above. In the simulation section, we show that the existing conditions for the measurement matrix, $\mathbf{A}$, do *not* apply to most of the routing matrices. Motivated by this observation, we aim to revisit these conditions and modify them so that they become more suitable to the network tomography problem. Then, we use linear program (LP) optimization to solve Eq. (1).

A. Expander Graphs

Definition 2. *A bipartite graph $G(X, Y, H)$ with a left degree d (i.e., $deg(v) = d \, \forall \, v \in X$) is a $(\phi, d, \epsilon) - expander$ if for any $\Phi \subset X$ with $|\Phi| \leq \phi$, the following condition holds:*

$$|N(\Phi)| \geq \ (1 - \epsilon)d \, |\Phi|, \tag{7}$$

where $N(\Phi)$ is a set of neighbors of Φ [4]. ϕ and ϵ are the "expansion factor" and the "error parameter," respectively.

Roughly speaking, in an expander graph, the degree of connectivity for a collection of nodes (with cardinality of up to ϕ) on the left-hand side (X) expands by that factor on the right-hand side (Y) [36]. Expander graphs are well-studied; authors in [37], [38], [39] show how to construct a $(\phi, d, \epsilon) - expander$ graph. In a key result, Berinde and Indyk in [40], [27] show that the bi-adjacency matrix of a $(2\phi, d, \epsilon) - expander$ graph can be used as the measurement matrix for a ϕ-sparse signal, for $\epsilon \leq \frac{1}{6}$.

The parameter ϵ in an expander graph is a design variable that is related to recovery error. The existing results require $\epsilon \leq 1/6$, which, as we will show, does not apply to most of the networks . In a network tomography problem, the measurement matrix is pre-determined, so we need to enlarge the bound on ϵ as high as possible to increase the likelihood that it leads to an identifiable network.

[4]Neighbors of Φ are nodes which are connected to at least one of the nodes in Φ.

The bipartite graph given in Figure 2 coordinates with the 1-identifiable network in Figure 1 with the routing matrix in Eq. (5). It is easy to see that this bipartite graph is an expander for $\epsilon = 1/4$. Motivated by this example, we relax the existing result for $\epsilon \leq 1/6$ to $\epsilon \leq 1/4$. In the simulation results (Section IV-B), we show that this relaxation increases the number of k-identifiable networks that satisfy the expansion property by 30%. In other words, for more than 30% of k-identifiable networks, we have $1/6 < \epsilon \leq 1/4$. For networks that satisfy the expansion property, LP optimization can be used to solve the tomography problem.

The analytical results are first derived for 1-identifiable networks because it is easier to give intuitive explanation. Then we generalized the result to k-identifiable networks with arbitrary k. In Section IV, we present simulation results to indicate that the proposed algorithm to recover a nearly $k \geq 1$-sparse vectors provides an acceptable estimation error.

B. 1-Identifiability

The following lemma provides an upper bound on the error of recovering $\mathbf{x}$ from its lower-dimensional projection $\mathbf{Ax}$ when $\mathbf{A}$ is a bi-adjacency matrix of a $(2, d, \epsilon)$-expander graph and $\epsilon \leq 1/4$.

lemma 1. *Let $\mathbf{A}$ be a bi-adjacency matrix of a $(2, d, \epsilon)$-expander graph with $\epsilon \leq 1/4$. Consider any two vectors, $\mathbf{x}$ and $\mathbf{x}'$, with the same projection under the measurement matrix $\mathbf{A}$, i.e., $\mathbf{Ax} = \mathbf{Ax}'$. Assume that $\mathbf{x}$ is 1-sparse. Further, without loss of generality, suppose that $\| \mathbf{x}' \|_1 \leq \| \mathbf{x} \|_1$. Let S be the set of the largest (in magnitude) elements of $\mathbf{x}$. Then,*

$$\| \mathbf{x}' - \mathbf{x} \|_1 \leq f(\epsilon) \| \mathbf{x}_{S^c} \|_1, \tag{8}$$

where

$$f(\epsilon) = \frac{2(1+2\epsilon)}{1-2\epsilon}, \quad \epsilon \leq \frac{1}{4}. \tag{9}$$

Proof: See Appendix.

The results from the previous lemma show that under some conditions, the link delay in a network may be estimated as the unique solution to Eq. (1). The following theorem relates the problem of delay estimation in a network $N(V,E)$ to results on expander graphs with $\epsilon \leq 1/4$ and shows that Eq. (1) can be solved for $\mathbf{x}$ using LP optimization.

Theorem 1. *Let $N(V,E)$ be a network with a set of paths $\mathcal{R}$ and a corresponding routing matrix $\mathbf{R}$. Suppose that $G(E,\mathcal{R},H)$ is a bipartite graph with bi-adjacency matrix $\mathbf{R}$. Assume that $\mathbf{x}$ is the true (unknown) delay vector of $N(V,E)$ and $\mathbf{y}=\mathbf{R}\mathbf{x}$ is the (given) end-to-end delay measurement. Let $\mathbf{x}'$ be a solution to the following LP optimization:*

$$\min \parallel \mathbf{x}' \parallel_1 \tag{10}$$

s.t.

$$\mathbf{R}\mathbf{x}' = \mathbf{y}.$$

Then

$$\parallel \mathbf{x}-\mathbf{x}' \parallel_1 \leq f(\epsilon) \parallel \mathbf{x}_{S^c} \parallel_1, \tag{11}$$

if G is a $(2,d,\epsilon)$-expander with $\epsilon \leq \frac{1}{4}$.

Proof: See Appendix.

If the routing matrix $\mathbf{R}$ is a bi-adjacency matrix of a $(2,d,\epsilon)$-expander graph ($\epsilon \leq 1/4$), then the equation $\mathbf{y}=\mathbf{R}\mathbf{x}$ has a unique solution for nearly 1-sparse delay vector $\mathbf{x}$ and it can be found using the LP optimization in Eq. (10). In the simulation results we will show that almost 70% of the networks are expanders with $\epsilon \leq 1/4$. In other words, for 70% of the networks the delay vector can be estimated using the LP optimization.

Note that if the true delay vector $\mathbf{x}$ is exactly 1-sparse (which almost never happens in practice because links always have nonzero delay), it implies that $\parallel \mathbf{x}_{S^c} \parallel_1 = 0$, which means that $\mathbf{x}'=\mathbf{x}$; i.e., l_1-norm minimization in Eq. (10) can recover $\mathbf{x}$ with zero estimation error. In other words, if the delay of all links in the network is zero except for maybe one link, the delay of that link can be exactly recovered from the end-to-end delay measurement. However, if the true delay vector contains links with small but nonzero delays (the more likely scenario), the estimation error is not zero and the above theorem yields an upper bound.

One of the conditions for expander graphs is d-regularity on the left-hand side. However, there exists some networks, $N(V,E)$, which are 1-identifiable, but their corresponding bipartite graphs are not d-regular. Hence, the result of Theorem 1 does not apply. An example of such a

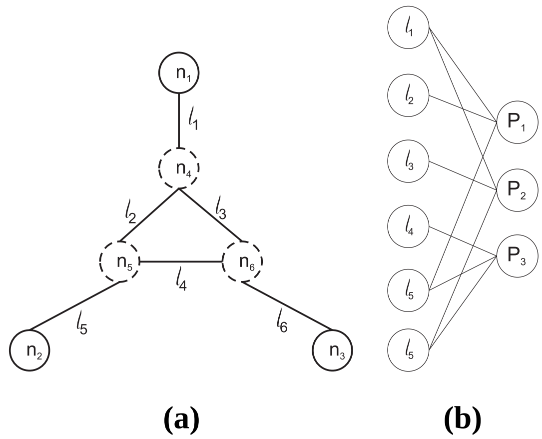


Fig. 3. A 1-identifiable network whose corresponding bipartite graph is not regular on the left side: (a) Network topology (b) Corresponding bipartite graph

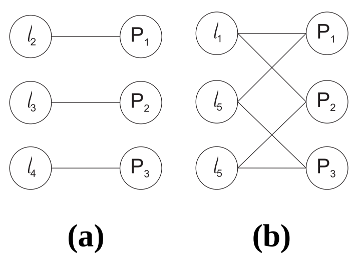


Fig. 4. Two subgraphs of the bipartite graph in Figure 3-b which are regular on their left side

network is depicted in Figure 3-(a) with the following routing matrix:

$$\mathbf{R} = \begin{array}{c} \\ P_1 : n_1 \rightsquigarrow n_2 \\ P_2 : n_1 \rightsquigarrow n_3 \\ P_3 : n_2 \rightsquigarrow n_3 \end{array} \left[\begin{array}{cccccc} l_1 & l_2 & l_3 & l_4 & l_5 & l_6 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{array} \right]. \tag{12}$$

The above routing matrix is a bi-adjacency matrix of the bipartite graph presented in Figure 3-(b). This bipartite graph is not regular in the left side because the degree of a node in the left set is either 1 or 2; hence, it cannot be an expander. However, Figures 4-(a) and (b), respectively, represent subgraphs of G with regular left degree 1 and 2; these subgraphs are expander graphs. The above observation convinces us that the result in Theorem 1 is extendable for networks whose corresponding bipartite graph is not regular (and is therefore not an expander) but can be partitioned into subgraphs that are expander graphs.

Theorem 2. *Let $N(V,E)$ be a network with routing matrix $\mathbf{R}$. Let $G(X,Y,H)$ be a bipartite graph with bi-adjacency matrix $\mathbf{R}$. Suppose that $G_i(X_i,Y,H_i)$, $i=1,2,...M$, are d_i-regular bipartite subgraphs of G such that*

- $X=\cup X_i$ *and* $X_i \cap X_j = \emptyset$ *for* $i \neq j$
- $H=\cup H_i$
- $d_i \neq d_j$ *for* $i \neq j$

Then, $N(V,E)$ is 1-identifiable if G_i is a $(2,d_i,\epsilon)$-expander graph with $\epsilon \leq \frac{1}{4}$, $i=1,2,\ldots,M$. Further, the link delay is the solution to LP optimization in Eq. (10).

Proof: See Appendix.

For future reference, we refer to the conditions in Theorem 2 as *1-identifiability expansion conditions*. If the network $N(V,E)$ satisfies these conditions, we refer to it as the 1-*identifiable expander network*.

Note that the 1-identifiable expansion conditions in Theorem 2 imply the following for any link pair l_i and l_j:

- They belong to different G_i's and hence have different degrees $d_i \neq d_j$.
- They belong to the same subgraph G_i, i.e., $d_i = d_j$. In that case, because G_i is a bipartite graph, they satisfy the expansion property in Eq. (7).

We state this observation formally in the following corollary.

Corollary 1. *Let $N(V,E)$ be a network with the routing matrix $\mathbf{R}$ and a set of paths $\mathcal{R}$. Let $G(E,\mathcal{R},H)$ be its corresponding bipartite graph with the bi-adjacency matrix $\mathbf{R}$. Then, one and only one of the following statements is true for any two links l_i and l_j in E, $i \neq j$:*

- $deg(l_i) > deg(l_j)$
- $deg(l_i) < deg(l_j)$
- $deg(l_i) + deg(l_j) - 4deg(l_i,l_j) \geq 0$

where $deg(l_i,l_j)$ is defined as the number of nodes connected to both l_i and l_j in the bipartite graph $G(E,\mathcal{R},H)$[5].

proof: See Appendix.

[5]It can also denote the number of paths going through both links l_i and l_j in the network $N(V,E)$ with the routing matrix $\mathbf{R}$.

C. k-identifiability

In this subsection, we extend our results to general k-identifiable networks that is defined as follows:

Definition 3. *A k-identifiable expander $N(V, E)$ is a network whose routing matrix $\mathbf{R}$ is the bi-adjacency matrix of a bipartite graph $G(X, Y, H)$ consisting of d_i-regular subgraphs $G(X_i, Y, H_i)$ with the following properties*

- $X = \cup X_i$ *and* $X_i \cap X_j = \emptyset$ *for* $i \neq j$
- $H = \cup H_i$
- $d_i \neq d_j$ *for* $i \neq j$
- $G(X_i, Y, H_i)$ *is a* $(2k, d_i, \epsilon)$*-expander with* $\epsilon \leq \frac{1}{4}$

The following theorem gives the expected estimation error when $l1$ optimization is used to recover links delay in a k-identifiable network.

Theorem 3. *Let $N(V, E)$ be a k-identifiable expander network with a set of paths $\mathcal{R}$ and a corresponding routing matrix $\mathbf{R}$. Assume that $\mathbf{x}$ is the true (unknown) delay vector of $N(V, E)$ and $\mathbf{y} = \mathbf{R}\mathbf{x}$ are the end-to-end delay measurements. Let $\mathbf{x}'$ be a solution to the following LP optimization:*

$$\min \parallel \mathbf{x}' \parallel_1 \tag{13}$$

$$\textit{s.t.}$$

$$\mathbf{R}\mathbf{x}' = \mathbf{y}.$$

Then

$$\mathbb{E}[\parallel \mathbf{x} - \mathbf{x}' \parallel_1] \leq f(\epsilon) \frac{1}{1 - \frac{k-1}{|\mathcal{R}|}} \parallel \mathbf{x}_{S^c} \parallel_1, \tag{14}$$

Proof: See Appendix.

First note that Eq. (14) reduces to Eq. (11) for $k = 1$. Second, for most well-designed wired network, we have $k \ll |\mathcal{R}|$ (number of congested links is much less than number of end-to-end paths in the network) and hence we have:

$$\mathbb{E}[\parallel \mathbf{x} - \mathbf{x}' \parallel_1] \leq f(\epsilon) \parallel \mathbf{x}_{S^c} \parallel_1 . \tag{15}$$

IV. Evaluation Results

In Section III, Theorem 3, we showed that if the routing matrix of a network is the bi-adjacency matrix of the union of disjoint expander graphs, that network is k-identifiable. Moreover, we can estimate internal link delay using an LP optimizer in Eq. (13). However, a legitimate 'big-picture' question arises: How many networks actually satisfy the conditions of Theorem 2; i.e., how many are k-identifiable expanders? In this section, we generate random Internet-type networks to study this question. Our simulation results show that our relaxation increases number of networks which satisfy expansion property by almost 30%.

For those networks that are k-identifiable expander–i.e., their routing matrix satisfies the condition in Definition 3–we determine the average normalized estimation error when there is k congested link in the network and show that the average normalized estimation error remains within an acceptable range. Next, we compare our algorithm with a recently developed delay tomography algorithm and show that the proposed algorithm yields a lower estimation error.

A. Generation of Networks with Random Topology

We use Inet version 3.0 [41], [42]– an Internet topology generator software (at AS[6] level)– to generate random graphs with the given power law and a fixed number of boundary nodes [7]. We create networks containing 5000 nodes with 5, 8, 10, 12, 16, and 20 boundary nodes, respectively. The output of Inet, which contains the set of neighbors of each node in the generated graph, is fed to matgraph toolbox in MATLAB [43] for modification. We first create a routing matrix containing the shortest paths between any boundary node pairs in the network. Then we delete all nodes and links that do not contribute to any of the above paths, since if a link is not covered by any end-to-end path, it is not identifiable. The remaining networks constitute our random set. In Figure 5, six examples of random networks are depicted.

B. Networks and Expansion Property

For the routing matrices of these random networks, we first examine how many of them satisfy the k-identifiability expansion conditions in Definition 3. For the network with a fixed number

[6]Autonomous System.

[7]Boundary nodes are nodes with degree one which act as injection points for probes in our problem.

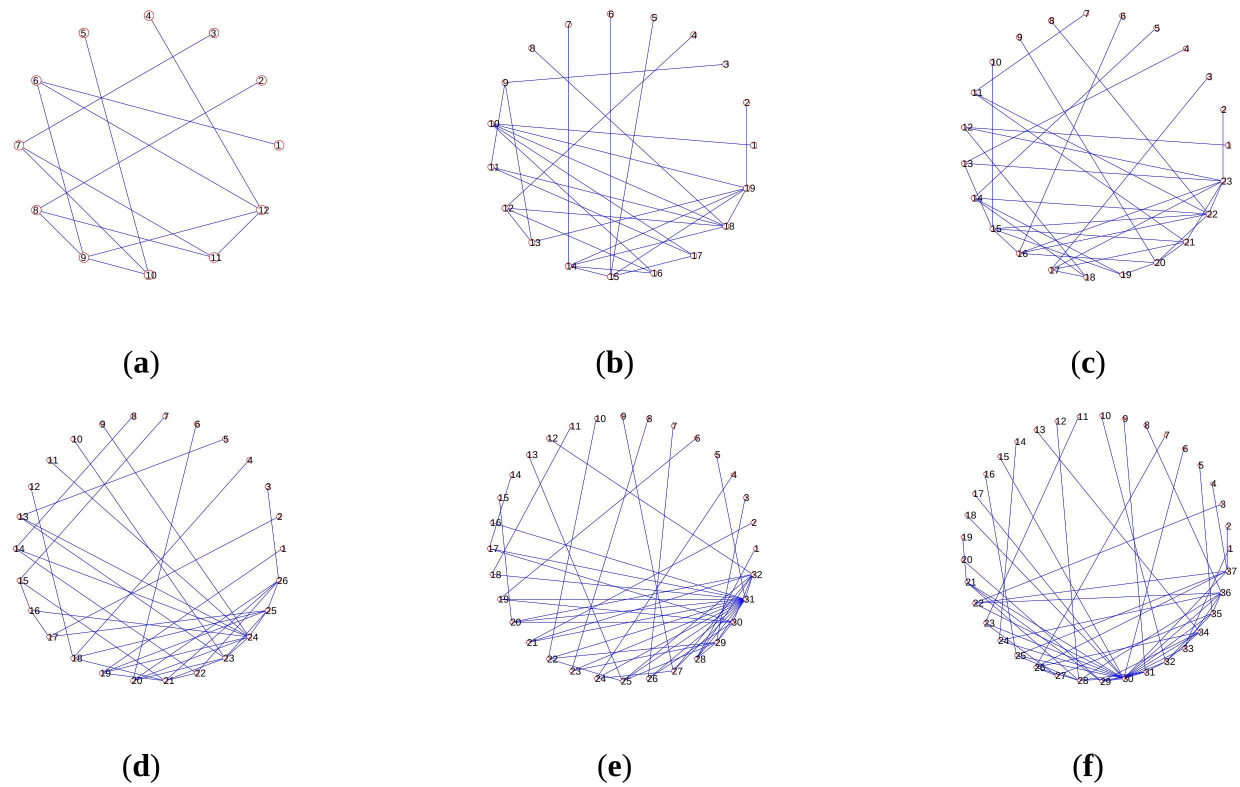


Fig. 5. Output of Inet after our modification in MATLAB with (a) 5 (b) 8 (c) 10 (d) 12 (e) 16 (f) 20 boundary nodes. Nodes with degree 1 represents injection nodes

of boundary nodes, fifty different topologies are created. Table I shows the percentage of them that satisfy the k-identifiability expansion property for $k = 1, 2, 3$.

To show the impact of our relaxation of ϵ in Lemma 1 and Theorem 3, in Table I, we also provide the percentage of networks that are k-identifiable, using $\epsilon \leq 1/6$. As one can see, by moving the bound on ϵ from $1/6$ to $1/4$, the number of networks satisfying the expansion property increases by almost 30%. In other words, 30% of k-identifiable networks are within $1/6 < \epsilon \leq 1/4$.

C. Delay Estimation: Simulation Experiments

Theorem 3 says that if the routing matrix of a network satisfies k-identifiable expander conditions, then link delays in the network can be estimated using Eq. (13). To examine the accuracy of the proposed delay estimation method, for each network created in Section IV-A, we calculate the average normalized estimation error for all links as follows. k reference links

TABLE I

FOR NETWORKS WITH FIXED NUMBER OF BOUNDARY NODES, COMPARING THE PERCENTAGE OF WHICH ARE K-IDENTIFIABLE EXPANDERS FOR $k = 1, 2, 3$ WITH $\epsilon \leq \frac{1}{4}$ AND $\epsilon \leq \frac{1}{6}$

		5	8	10	12	16	20
$k = 1$	$\epsilon \leq \frac{1}{4}$	80%	82%	76%	72%	74%	72%
	$\epsilon \leq \frac{1}{6}$	38%	52%	42%	40%	32%	30%
$k = 2$	$\epsilon \leq \frac{1}{4}$	0%	60%	62%	56%	50%	50%
	$\epsilon \leq \frac{1}{6}$	0%	0%	30%	20%	24%	22%
$k = 3$	$\epsilon \leq \frac{1}{4}$	0%	0%	0%	56%	50%	46%
	$\epsilon \leq \frac{1}{6}$	0%	0%	0%	0%	0%	22%

are selected and assigned a delay of 10 ms to denote congestion, $k = 1, 2, 3$. All other links in the network are assumed to experience i.i.d exponentially distributed delays with average μ, i.e.,

$$f_l(t) = \frac{1}{\mu} exp(-\frac{t}{\mu}) \quad \forall l \in E, \tag{16}$$

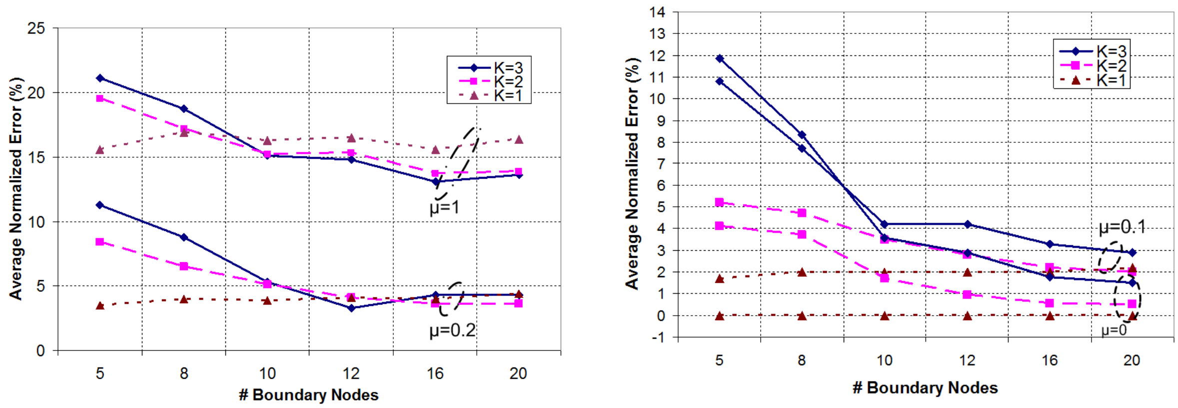
where $f_l(t)$ is the delay for link l and $\mu \in [0, 1]$ to denote that these links do not undergo congestion.

We exploit the proposed LP optimization in (10) to estimate link delays. For the network, the normalized estimation error for each congested link inside the network is calculated as follows:

$$norm.\ err = \frac{\| \mathbf{x} - \hat{\mathbf{x}} \|_2}{\| \mathbf{x} \|_2}, \tag{17}$$

where $\mathbf{x}$ and $\hat{\mathbf{x}}$ are the true and estimated delay vectors respectively.

Figure 6 presents the average normalized estimation error when there are k congested links inside the network for $k = 1, 2, 3$ and LP optimization Eq. (10) is used to estimate the delay. As expected, the average normalized estimation error for different μ (vector $\mathbf{x}$ is nearly k-sparse) mimics the expected trend from Eq. (14). An interesting observation in in Figure 6 is the fact that for $k > 1$, the average normalized estimation error has a decreasing trend and for large networks, it is almost the same as in $k = 1$ case. The reason is that, as one can infer from Table I, the probability of being a k-identifiable expander is higher for large networks than a small ones for $k = 2, 3$. This is acceptable because for small networks, with a few number of links, probability of having more than one congested link is negligible.



(a) **(b)**

Fig. 6. Average normalized estimation error in networks satisfying conditions given in Theorem 2 when there are k deficient link within the network for different average delay μ in Eq. (16).

D. Delay Estimation: Cumulative Distribution Function

In this section, we compare our results with those produced by the CF-estimator, one of the recent and novel delay estimators proposed in [28]. We provide the cumulative probability of the normalized estimation error using both methods and show that the proposed method provides less probability of error.

Let $\mathbf{x}$ be the actual delay of the links and $\hat{\mathbf{x}}$ be the output of the delay estimator. We aim to calculate the following CDF:

$$P(\frac{\parallel \mathbf{x} - \hat{\mathbf{x}} \parallel_2}{\parallel \mathbf{x} \parallel_2} \leq \delta). \tag{18}$$

Figure 7 presents the CDF of the normalized estimation error when there are k congested links inside the network for $k = 1, 2, 3$ and $\delta \in [0, .5]$. For each CDF, 200 random networks[8] are generated and for each generated network, link delays are assigned as we describe in Section IV-C. As one can see, the proposed algorithm outperforms the CF algorithm due to the fact that it uses sparsity information.

As one can see the proposed algorithm outperforms CF algorithm. The reason is that it uses sparsity as its side information. It is worth mentioning again that to provide QoS in a network, either it is for multimedia streaming, gaming or any other live application, links with high delays

[8] $\mu \in \{0, .1, .2, 1\}$.

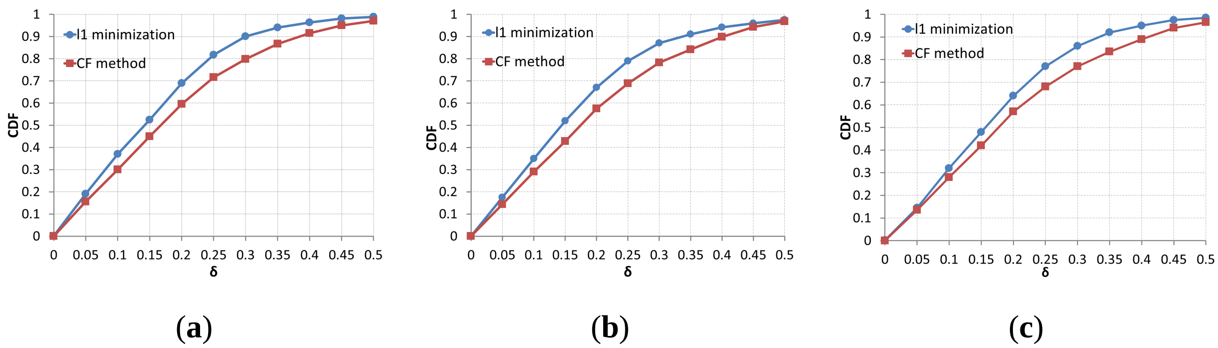


Fig. 7. Cumulative distribution function of estimation error for (a)$k = 1$ (b)$k = 2$ (c)$k = 3$. The proposed algorithm outperforms CF method.

are more important. Hence the l_1 minimization given in Eq. (13) focus on finding the k links with the highest delays and it results in a better estimation.

V. Conclusion

In this manuscript, we investigate the application of expander graphs and compressed sensing to network tomography. As shown in the paper by examples and simulation evidence, the current results on expander graphs do not apply to most of the networks. Hence, we modify some of the results to be more suitable for the delay estimation problem. We show that the number of Internet-topology-based networks satisfying new conditions is increased by 30%. For those networks, we compare delay estimation based on compressed sensing (proposed algorithm) with one of the state-of-the-art delay estimation algorithms in the literature. The simulation results show that compressed sensing provides better estimation, i.e., less estimation error.

Most of the network parameters such as link delay are, in fact, non-negative numbers, and by using this information, it would be better to estimate links' statuses. There are some works in the literature to recover non-negative signals [23], and one of our future works is to apply these theories to delay estimation.

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APPENDIX A
PROOFS OF THEOREMS

Lemma 1:

We first proof the following lemma which characterizes the null space of bi-adjacency matrix of an expander graph and will be used to bound the error in the recovery of $\mathbf{x}$ from its compressed projection $\mathbf{y}$.

lemma 2. *Let $G(V_1, V_2, E)$ be a $(2, d, \epsilon)$-expander with $\epsilon \leq 1/4$ and $\mathbf{A}_{m\times n}$ be its bi-adjacency matrix. Assume $\mathbf{w}$ lies in the null space of $\mathbf{A}$ (i.e., $\mathbf{A}\mathbf{w} = \mathbf{0}$) and let S be any singleton set of coordinates of the $\mathbf{w}$, i.e., $S = \{i\}$, $i \in \{1, ..., n\}$. Then*

$$\| \mathbf{w}_S \|_1 \leq 2\epsilon \| \mathbf{w}_{S^c} \|_1 \ . \tag{19}$$

Proof:

Let $\mathbf{A}'$ be the submatrix of $\mathbf{A}$ containing rows from $N(S)$. Since $|S| = 1$ and graph is left d-regular, $\| \mathbf{A}'\mathbf{w}_S \|_1 = \| \mathbf{A}\mathbf{w}_S \|_1 = d \| \mathbf{w}_S \|_1$. We have

$$\begin{aligned} 0 = \| \mathbf{A}'\mathbf{w} \|_1 &= \| \mathbf{A}'\mathbf{w}_S + \mathbf{A}'\mathbf{w}_{S^c} \|_1 \\ &\geq \| \mathbf{A}'\mathbf{w}_S \|_1 - \| \mathbf{A}'\mathbf{w}_{S^c} \|_1 \\ &= d \| \mathbf{w}_S \|_1 - \| \mathbf{A}'\mathbf{w}_{S^c} \|_1 \ . \end{aligned} \tag{20}$$

Each set of two nodes in the left part has at least $2(1-\epsilon)d$ neighbor nodes on the right side (expansion definition). Since each node at the left has degree d, number of common nodes on the right hand side [9] is at most $2d - 2(1-\epsilon)d = 2\epsilon d$. That means each column of $\mathbf{A}'$ (except the one corresponding to S) has at most $2\epsilon d$ number of "1"s, yielding,

$$\| \mathbf{A}'\mathbf{w}_{S^c} \|_1 \leq 2\epsilon d \| \mathbf{w}_{S^c} \|_1 \ . \tag{21}$$

Therefore

$$0 \geq d \| \mathbf{w}_S \|_1 - 2\epsilon d \| \mathbf{w}_{S^c} \|_1, \tag{22}$$

which means

$$\| \mathbf{w}_S \|_1 \leq 2\epsilon \| \mathbf{w}_{S^c} \|_1 \ . \tag{23}$$

[9] Nodes on the right which are connected to both nodes on the left hand side.

The above argument is valid if G is a $(2, d, \epsilon)$-expander graph. We now show that it implies $\epsilon \leq 1/4$. let Φ be a set of any two nodes on the left hand side of the $(2, d, \epsilon)$-graph. By the definition of expander graphs we have:

$$N(\Phi) \geq 2(1-\epsilon)d. \tag{24}$$

If the two nodes in Φ are connected to exactly the same nodes on the right hand side, there is no way to distinguish between them. In other words, if the two nodes are connected to the same nodes on the right hand side, the bi-adjacency matrix $\mathbf{A}$ is rank deficient. To avoid that [10] we need $N(\Phi)$ to be strictly greater than d, i.e., $N(\Phi) \geq d+1$. That means $2(1-\epsilon)d$ must be at least $d+1$. Hence, we have the following upper bound on ϵ

$$\epsilon \leq \frac{d-1}{2d}. \tag{25}$$

For an upper bound that is independent of d, we find the infimum of right hand side and choose ϵ to satisfy that case. Clearly we have

$$\inf_{d=2,3,4\ldots} \frac{d-1}{2d} = \frac{1}{4}, \tag{26}$$

implying that $\epsilon \leq \frac{1}{4}$[11].

□

Now, let vector $\mathbf{w}$ be in null space of $\mathbf{A}$, i.e., $\mathbf{w} \in \mathcal{N}(\mathbf{A})$. Using Eq. (8) we have:

$$\begin{aligned} \| \mathbf{w}_S \|_1 &\leq 2\epsilon \| \mathbf{w}_{S^c} \|_1 \\ \| \mathbf{w}_S \|_1 + 2\epsilon \| \mathbf{w}_S \|_1 &\leq 2\epsilon \| \mathbf{w}_{S^c} \|_1 + 2\epsilon \| \mathbf{w}_S \|_1 \\ \| \mathbf{w}_S \|_1 &\leq \frac{2\epsilon}{1+2\epsilon} \| \mathbf{w} \|_1 . \end{aligned} \tag{27}$$

[10]and also to satisfy the concept of expansion,

[11]Note that we exclude case of $d = 1$, since each bipartite graph which is left 1-regular and $N(\Phi) \geq 2$ is an expander graph.

Now, let $\mathbf{y} = \mathbf{x}' - \mathbf{x}$. Clearly $\mathbf{y} \in \mathcal{N}(\mathbf{A})$ and we have:

$$\begin{aligned}
\| \mathbf{x} \|_1 &\geq \| \mathbf{x}' \|_1 \\
&= \| (\mathbf{x} + \mathbf{y})_S \|_1 + \| (\mathbf{x} + \mathbf{y})_{S^c} \|_1 \\
&= \| \mathbf{x}_S + \mathbf{y}_S \|_1 + \| \mathbf{x}_{S^c} + \mathbf{y}_{S^c} \|_1 \\
&\geq \| \mathbf{x}_S \|_1 - \| \mathbf{y}_S \|_1 + \| \mathbf{y}_{S^c} \|_1 - \| \mathbf{x}_{S^c} \|_1 \\
&= \| \mathbf{x} \|_1 - 2 \| \mathbf{x}_{S^c} \|_1 + \| \mathbf{y} \|_1 - 2 \| \mathbf{y}_S \|_1 \\
&\geq \| \mathbf{x} \|_1 - 2 \| \mathbf{x}_{S^c} \|_1 + (1 - \frac{4\epsilon}{1+2\epsilon}) \| \mathbf{y} \|_1,
\end{aligned} \tag{28}$$

where in the last equality, Eq. (27) is used. Therefore we have:

$$\| \mathbf{x}' - \mathbf{x} \|_1 = \| \mathbf{y} \|_1 \leq f(\epsilon) \| \mathbf{x}_{S^c} \|_1, \tag{29}$$

where $f(\epsilon) = \frac{2(1+2\epsilon)}{1-2\epsilon}$.

■

Theorem 1:

Let $\mathbf{x}'$ be the solution to optimization problem in Eq. (19). It means $\mathbf{R}\mathbf{x}' = \mathbf{R}\mathbf{x}$ and $\| \mathbf{x}' \|_1 \leq \| \mathbf{x} \|_1$. On the other hand, G is a $(2, d, \epsilon)$-expander graph with the bi-adjacency matrix $\mathbf{R}$. Consequently Eq. (9) in Theorem 1 holds for $\mathbf{x}$ and $\mathbf{x}'$.

■

Theorem 2:

We prove the theorem for the case in which $G(X, Y, H)$ has only two expander subgraphs. The general case can be easily extended following the same way. Let $G_1(X_1, Y, H_1)$ with $|X_1| = m$, and $G_2(X_2, Y, H_2)$ with $|X_2| = n - m$, be two d_i-regular ($d_1 \neq d_2$) subgraphs of $G(X, Y, H)$ with bi-adjacency matrices $\mathbf{R}_1$ and $\mathbf{R}_2$, respectively. Without lost of generality, let rename the elements in X such that $\mathbf{R} = [\mathbf{R}_1 \, \mathbf{R}_2]$.

Now suppose $\mathbf{w} \in \mathbb{R}^n$ belong to null space of $\mathbf{R}$; $\mathbf{w} = [\mathbf{w}_1^t \mathbf{w}_2^t]^t$.

Let S be any set of $k = 1$ coordinates of $\mathbf{w}$. Further, let $\mathbf{R}'$ be submatrix of $\mathbf{R}$ containing rows from $N(S)$. We consider two following cases:

Case 1: $S \subset \{1, 2, ..., m\}$:

In this case S represents a node in $G_1(X_1, Y, H_1)$ which is a $(2, d_1, \epsilon)$-expander by assumption. Similar to proof of Lemma 1, $\parallel \mathbf{R}'\mathbf{w}_S \parallel_= \parallel \mathbf{R}\mathbf{w}_S \parallel_1 = d_1 \parallel \mathbf{w}_S \parallel_1$. We have

$$\begin{aligned} 0 = \parallel \mathbf{R}'\mathbf{w} \parallel_1 &= \parallel \mathbf{R}'\mathbf{w}_S + \mathbf{R}'\mathbf{w}_{S^c} \parallel_1 \\ &\geq \parallel \mathbf{R}'\mathbf{w}_S \parallel_1 - \parallel \mathbf{R}'\mathbf{w}_{S^c} \parallel_1 \\ &= d_1\mathbf{w}_S - \parallel \mathbf{R}'\mathbf{w}_{S^c} \parallel_1 . \end{aligned} \tag{30}$$

Since G_1 is left d_1-regular $\mathbf{R}'$ has d_1 rows. Using that, we can put an upper bound on $\parallel \mathbf{R}'\mathbf{w}_{S^c} \parallel_1$ as follows. Let $\mathbf{r}_i^t$ be the i-th rows of $\mathbf{R}'$. Then

$$\begin{aligned} \parallel \mathbf{R}'\mathbf{w}_{S^c} \parallel_1 &= \sum_{i=1}^{d_1} |\mathbf{r}_i^t \mathbf{w}_{S^c}| \\ &= \sum_{i=1}^{d_1} |\sum_{j=1}^{|X|} r_{ij} w_{S^c j}| \\ &\overset{(1)}{\leq} \sum_{j=1}^{d_1} \sum_{i=1}^{|X|} r_{ij} |w_{S^c j}| \\ &= \sum_{j=1}^{|X|} \sum_{i=1}^{d_1} r_{ij} |w_{S^c j}| \\ &= \sum_{j=1}^{|X|} |w_{S^c j}| \sum_{i=1}^{d_1} r_{ij} \\ &= \sum_{j=1}^{|X_1|} |w_{S^c j}| \sum_{i=1}^{d_1} r_{ij} + \sum_{j=|X_1|+1}^{|X|} |w_{S^c j}| \sum_{i=1}^{d_1} r_{ij}, \end{aligned} \tag{31}$$

where for inequality (1), we used the triangular inequality and the fact that $r_{ij} \in \{0, 1\}$. Since $G_1(X_1, Y, H_1)$ is an $(2, d_1, \epsilon)$-expander, each two nodes at the right hand side have at most $2\epsilon d_1$ neighbors at the right in common. That means, each column in $\mathbf{R}'$ has at most $2\epsilon d$, i.e., $\sum_{i=1}^{d_1} r_{ij} \leq 2\epsilon d_1$ for each $i = \{1, 2, ..., |X_1|\}$. On the other hand since $\mathbf{R}'$ has d_1 rows, $\sum_{i=1}^{d_1} r_{ij} \leq$

d_1 for each $i = \{|X_1|+1, 2, ..., |X|\}$. Using these facts and result in (31) we get:

$$\begin{aligned} \| \mathbf{R}'\mathbf{w}_{S^c} \|_1 &\leq \sum_{j=1}^{|X_1|} \big(|w_{S^cj}| \sum_{i=1}^{d_1} r_{ij}\big) + \sum_{j=|X_1|+1}^{|X|} \big(|w_{S^cj}| \sum_{i=1}^{d_1} r_{ij}\big) \\ &\leq 2\epsilon d_1 \sum_{j=1}^{|X_1|} |w_{S^cj}| + d_1 \sum_{j=|X_1|+1}^{|X|} |w_{S^cj}| \\ &= 2\epsilon \| \mathbf{w}_{1S^c} \|_1 + d_1 \| \mathbf{w}_2 \| . \end{aligned}$$

Substituting above inequality in (30) we have:

$$0 \geq d_1 \| \mathbf{w}_S \|_1 - 2\epsilon d_1 \| \mathbf{w}_{1S^c} \|_1 - d_1 \| \mathbf{w}_2 \|_1 . \tag{32}$$

Therefore we have the following upper bound

$$\| \mathbf{w}_S \|_1 \leq 2\epsilon \| \mathbf{w}_{1S} \|_1 + \| \mathbf{w}_2 \|_1, \tag{33}$$

which also can be written as below:

$$\| \mathbf{w}_S \|_1 \leq \frac{2\epsilon}{1+2\epsilon} \| \mathbf{w}_1 \|_1 + \frac{1}{1+2\epsilon} \| \mathbf{w}_2 \| . \tag{34}$$

Case 2: $S \subset \{m+1, 2, ..., n\}$:

By the same argument as Case 1, we have:

$$\begin{aligned} 0 = \| \mathbf{R}\mathbf{w} \|_1 &= \| \mathbf{R}'\mathbf{w} \|_1 \\ &\geq d_2 \| \mathbf{w}_S \|_1 - \| \mathbf{R}'\mathbf{w}_{S^c} \|_1 . \end{aligned} \tag{35}$$

As in case 1, we can put an upper bound on $\| \mathbf{R}'\mathbf{w}_{S^c} \|_1$ as follows.

$$\begin{aligned} \| \mathbf{R}'\mathbf{w}_{S^c} \|_1 &\leq \sum_{j=1}^{|X_1|} \big(|w_{S^cj}| \sum_{i=1}^{d_2} r_{ij}\big) + \sum_{j=|X_1|+1}^{|X|} \big(|w_{S^cj}| \sum_{i=1}^{d_2} r_{ij}\big) \\ &\leq d_1 \sum_{j=1}^{|X_1|} |w_{S^cj}| + 2\epsilon d_2 \sum_{j=|X_1|+1}^{|X|} |w_{S^cj}| \\ &\leq d_1 \| \mathbf{w}_1 \|_1 + 2\epsilon d_2 \| \mathbf{w}_{2S^c} \|_1 . \end{aligned}$$

Using above inequality and results from Eq. (35), we have the following upper bound for $\| \mathbf{w}_S \|_1$.

$$\| \mathbf{w}_S \|_1 \leq 2\epsilon \| \mathbf{w}_{2S^c} \|_1 + \| \mathbf{w}_1 \|_1, \tag{36}$$

which also can be written as below:

$$\| \mathbf{w}_S \|_1 \leq \frac{2\epsilon}{1+2\epsilon} \| \mathbf{w}_2 \|_1 + \frac{1}{1+2\epsilon} \| \mathbf{w}_1 \| . \tag{37}$$

Now suppose for a given $\mathbf{y}$, two 1-sparse vectors $\mathbf{u}$ and $\mathbf{v}$ satisfies equality $\mathbf{y} = \mathbf{R}\mathbf{u} = \mathbf{R}\mathbf{v}$. Let $\mathbf{w} = \mathbf{u} - \mathbf{v}$ and $\mathbf{u} = [\mathbf{u}_1^t \mathbf{u}_2^t]^t$, $\mathbf{v} = [\mathbf{v}_1^t \mathbf{v}_2^t]^t$ and $\mathbf{w} = [\mathbf{w}_1^t \mathbf{w}_2^t]^t$. Clearly, the following equalities hold:

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{u}_1 - \mathbf{v}_1, \\ \mathbf{w}_2 &= \mathbf{u}_2 - \mathbf{v}_2. \end{aligned} \tag{38}$$

Without loss of generality, let's assume $\| \mathbf{u} \|_1 \geq \| \mathbf{v} \|_1$. We consider two following cases:

Case 1: $S \subset \{1, 2, ..., m\}$

$$\begin{aligned} \| \mathbf{u} \|_1 &\geq \| \mathbf{v} \|_1 \\ &= \| \mathbf{u}_1 + \mathbf{w}_1 \|_1 + \| \mathbf{u}_2 + \mathbf{w}_2 \|_1 \\ &= \| \mathbf{u}_{1S} + \mathbf{w}_{1S} \|_1 + \| \mathbf{u}_{1S^c} + \mathbf{w}_{1S^c} \|_1 + \\ &\quad \| \mathbf{u}_2 + \mathbf{w}_2 \|_1 \\ &\geq \| \mathbf{u}_{1S} \|_1 - \| \mathbf{w}_{1S} \|_1 + \| \mathbf{w}_{1S^c} \|_1 - \| \mathbf{u}_{1S^c} \|_1 + \\ &\quad \| \mathbf{w}_2 \|_1 - \| \mathbf{u}_2 \|_1 \\ &= \| \mathbf{u}_{1S} \|_1 - (\| \mathbf{u}_{1S^c} \|_1 + \| \mathbf{u}_2 \|_1) + \\ &\quad (\| \mathbf{w}_{1S^c} \|_1 + \| \mathbf{w}_2 \|_1) - \| \mathbf{w}_{1S} \|_1 \end{aligned} \tag{39}$$

Since $S \subset \{1, 2, ..., m\}$, we have $\| \mathbf{w}_2 \|_1 + \| \mathbf{w}_{1S^c} \|_1 = \| \mathbf{w}_{S^c} \|_1$ and $\| \mathbf{u}_2 \|_1 + \| \mathbf{u}_{1S^c} \|_1 = \| \mathbf{u}_{S^c} \|_1$. So Eq. (39) can be simplified as bellow:

$$\begin{aligned} 2 \| \mathbf{u}_{S^c} \|_1 &\geq \| \mathbf{w}_{S^c} \|_1 - \| \mathbf{w}_{1S} \|_1 \\ &= \| \mathbf{w} \|_1 - 2 \| \mathbf{w}_{1S} \|_1 . \end{aligned} \tag{40}$$

by using Eq. (34) we have:

$$2 \| \mathbf{u}_{S^c} \|_1 \geq \frac{1 - 2\epsilon}{1 + 2\epsilon} \| \mathbf{w_1} \|_1 - \frac{1 - 2\epsilon}{1 + 2\epsilon} \| \mathbf{w_2} \|_1 . \tag{41}$$

By Eq. (38) and triangular inequality we have:

$$\| \mathbf{w}_2 \|_1 \leq \| \mathbf{u}_2 \|_1 + \| \mathbf{v}_2 \|_1 . \tag{42}$$

Applying above inequality to (41) we have:

$$\frac{1+2\epsilon}{1-2\epsilon}\left[2\,\|\mathbf{u}_{1S^c}\|_1+\frac{1-2\epsilon}{1+2\epsilon}\big(\|\mathbf{u}_2\|_1+\|\mathbf{v}_2\|_1\,\big)\right]\geq\|\mathbf{w}\|_1\,. \tag{43}$$

Clearly $\| \mathbf{u}_{S^c} \|_1 \geq \| \mathbf{u}_{1S^c} \|_1$, $\| \mathbf{u}_{S^c} \|_1 \geq \| \mathbf{u}_2 \|_1$ and $\| \mathbf{v}_{S^c} \|_1 \geq \| \mathbf{v}_2 \|_1$. Therefore, the following inequalities hold:

$$\frac{3+2\epsilon}{1-2\epsilon} \| \mathbf{u}_{S^c} \|_1 + \| \mathbf{v}_{2S^c} \|_1 \geq \| \mathbf{w} \|_1 \,. \tag{44}$$

Now let $j \in S^c$. There is a path p^* which goes through link j and not link in S (since it is a logical network). Let $\mathbf{r}_{p^*}$ be the corresponding row for p^* in routing matrix $\mathbf{R}$. Since $\mathbf{Ru} = \mathbf{Rv}$, we have:

$$\mathbf{r}_{p^*}\mathbf{u} = \mathbf{r}_{p^*}\mathbf{v} \geq v_j. \tag{45}$$

Since p^* doesn't go through link S, its corresponding entry in $\mathbf{r}_{p^*}$ is zero. Hence we have $\| \mathbf{u}_{S^c} \|_1 \geq \mathbf{r}_{p^*}\mathbf{u}$. Therefore we can have the following upper bound for every entry of $v_j \; \forall j \in S^c$

$$\| \mathbf{u}_{S^c} \|_1 \geq v_j \; \forall j \in S^c. \tag{46}$$

By adding up both side of the inequality for all $j \in S^c$ we have:

$$|S^c| \; \| \mathbf{u}_{S^c} \|_1 \geq \| \mathbf{v}_{S^c} \|_1 \,. \tag{47}$$

Clearly $|X| > |S^c|$. Therefore, the following upper bound is valid for $\mathbf{w} = \mathbf{u} - \mathbf{v}$:

$$(\frac{3+2\epsilon}{1-2\epsilon} + |X|) \; \| \mathbf{u}_{S^c} \|_1 \geq \| \mathbf{w} \|_1 \,. \tag{48}$$

Case 2: $S \subset \{m+1, 2, ..., n\}$. By the same argument as case 1 we have:

$$(\frac{3+2\epsilon}{1-2\epsilon} + |X|) \; \| \mathbf{u}_{S^c} \|_1 \geq \| \mathbf{w} \|_1 \,. \tag{49}$$

Note that set X in $G(X, Y, H)$ is the same as E in network $N(V, E)$. The rest of the proof for LP optimization is the same as Theorem 1.

■

Proof of Corollary 1:

$N(V, E)$ with the routing matrix $\mathbf{R}$ is a 1-identifiable expander. By definition 3, that means bipartite graph $G(X, Y, H)$ with the bi-adjacency matrix $\mathbf{R}$ is a union of left d_i-regular bipartite graphs, $G(X_i, Y, H_i)$ such that:

- $X = \cup X_i$ and $X_i \cap X_j = \emptyset$ for $i \neq j$
- $H = \cup H_i$
- $d_i \neq d_j$ for $i \neq j$
- $G(X_i, Y, H_i)$ is a $(2, d_i, \epsilon)$-expander with $\epsilon \leq \frac{1}{4}$

Now let consider two links l_i and l_j. If $deg(l_i) \neq deg(l_j)$, then clearly one of the first two statements in Corollary 4 would be true. If $deg(l_i) = deg(l_j) = d_i$ they belong to the same subgraph, say, $G(X_i, Y, H_i)$. By the last condition in Definition 3, $G(X_i, Y, H_i)$ is a $(2, d_i, \epsilon)$-expander. Now let $\Phi = \{l_i, l_j\}$. By the definition of the expander graphs in Definition 2, the following holds for Φ:

$$|N(\Phi)| \geq (1-\epsilon)d|\Phi| = 2(1-\epsilon)d. \tag{50}$$

By Theorem 2 maximum value possible for ϵ is $\frac{1}{4}$. Therefore minimum value for the right hand side of the above inequality would be achieved if $\epsilon = \frac{1}{4}$ and that is:

$$|N(\Phi)| \geq \frac{3}{2}d. \tag{51}$$

$deg(l_i, l_j)$ is defined to be number of nodes connected to both l_i and l_j. We can calculate total number of nodes connected to at least one of l_i and l_j as follows:

$$|N(\Phi)| = deg(l_i) + deg(l_j) - deg(l_i, l_j) = 2d - deg(l_i, l_j). \tag{52}$$

Substituting above equality in Eq. (51) results in

$$d - 2deg(l_i, l_j) \geq 0, \tag{53}$$

which can also be written as follows:

$$2d - 2deg(l_i, l_j) = deg(l_i) + deg(l_j) - 4deg(l_i, l_j) \geq 0. \tag{54}$$

■

proof of Theorem 3:

We first assume that $\mathbf{R}$ is a bi-adjacency matrix of a $(2k, \epsilon, d)$-expander graph and prove that following lemma which characterizes the null space of $\mathbf{R}$.

Lemma 3: Assume $\mathbf{w}$ lies in the null space of $\mathbf{R}_{r\times n}$ (i.e., $\mathbf{AR} = \mathbf{0}$) and let S be any set of coordinates of the $\mathbf{w}$ with $|S| \leq k$, $S \subset \{1, ..., n\}$. Then

$$\mathbb{E}[\| \mathbf{w}_S \|_1] \leq \frac{2\epsilon}{1 - \frac{k-1}{r}} \| \mathbf{w}_{S^c} \|_1 . \tag{55}$$

Proof:

Let $\mathbf{R}'$ be the submatrix of $\mathbf{R}$ containing rows from $N(S)$. By definition of $\mathbf{w}_S$, we have $\| \mathbf{R}\mathbf{w}_S \|_1 = \| \mathbf{R}'\mathbf{w}_S \|_1$. Hence,

$$\begin{aligned} 0 = \| \mathbf{R}'\mathbf{w} \|_1 &= \| \mathbf{R}'\mathbf{w}_S + \mathbf{R}'\mathbf{w}_{S^c} \|_1 \\ &\geq \| \mathbf{R}'\mathbf{w}_S \|_1 - \| \mathbf{R}'\mathbf{w}_{S^c} \|_1 \\ &= \boldsymbol{\beta} \| \mathbf{w}_S \|_1 - \| \mathbf{R}'\mathbf{w}_{S^c} \|_1, \end{aligned} \tag{56}$$

where $\boldsymbol{\beta}$ is a random variable showing the number of paths that uniquely belong to only one of the k links. Now let take the expected value from the both side of inequality (56) and we drive:

$$\begin{aligned} 0 &\geq \mathbb{E}[\boldsymbol{\beta} \| \mathbf{w}_S \|_1 - \| \mathbf{A}'\mathbf{w}_{S^c} \|_1] \\ &= \mathbb{E}[\boldsymbol{\beta} \| \mathbf{w}_S \|_1] - \mathbb{E}[\| \mathbf{A}'\mathbf{w}_{S^c} \|_1]] \\ &= \mathbb{E}[\boldsymbol{\beta}]\mathbb{E}[\| \mathbf{w}_S \|_1] - \mathbb{E}[\| \mathbf{A}'\mathbf{w}_{S^c} \|_1]. \end{aligned} \tag{57}$$

The last equality is valid because $\boldsymbol{\beta}$ is independent of null space $\mathbf{R}$. Next, we will calculate $\mathbb{E}[\boldsymbol{\beta}]$.

Assume a particular path P goes over one of the links. This means that in the bipartite graph with bi-adjacency matrix $mathbf R$, P belongs to one of the links. The probability that it does not belong to any other $k-1$ links is $(1 - \frac{1}{|\mathcal{R}|})^{k-1}$. Since there are k links and each belong to d paths (recall that the lemma assumes d-regularity) there must be dk total paths where some of them are common between different links. On the other hand, each path uniquely belongs to one of the links with probability $(1 - \frac{1}{|\mathcal{R}|})^{k-1}$. Therefore, $\boldsymbol{\beta}$ has a binomial distribution and hence the average number of unique paths can be calculated as:

$$\mathbb{E}[\boldsymbol{\beta}] = dk(1 - \frac{1}{|\mathcal{R}|})^{k-1} \approx dk(1 - \frac{k-1}{|\mathcal{R}|}), \tag{58}$$

where the approximation is from the first two terms of the Taylor series and it is valid when $|\mathcal{R}| \gg 1$.

By the same argument as in proof of lemma 2, we have $\| \mathbf{A}'\mathbf{w}_{S^c} \|_1 = 2kd\epsilon \| \mathbf{w}_{S^c} \|_1$. By substituting these recent findings in Eq. (56), we can derive the following inequality:

$$\mathbb{E}[\| \mathbf{w}_S \|] \leq \frac{2\epsilon}{1 - \frac{k-1}{|\mathcal{R}|}} \| \mathbf{w}_{S^c} \|_1 \tag{59}$$

The above inequality is similar to Eq. (23), except for the expected value part; i.e. $\mathbb{E}$. The rest of the proof is similar to proof of Theorem 1 and Theorem 2 by substituting $\| \mathbf{x} - \mathbf{x}' \|_1$ with $\mathbb{E}[\| \mathbf{x} - \mathbf{x}' \|_1]$.

■