Imaging of a Target Through Random Media Using a Short-pulse Focused Beam

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***Abstract*—This paper presents a general formulation of a theory of imaging through random obscuring layers. Previously we presented a theory for the temporal behavior of a short pulse scattered from a random medium and from a point target. In this paper, we generalize our study to include the imaging of objects of finite size and the actual imaging pattern at the receiver. This involves the study of two-frequency fourth order moments. Numerical examples are given to illustrate several important features, including the optical depth, backscattering enhancement, shower curtain effects, aperture size, bandwidth and target size.**

***Index Terms*— imaging, random media, pulse, focused beam**

I. Introduction

IMAGING through random layers has attracted considerable attention in recent years [1,2]. In particular, OCT (Optical Coherence Tomography) makes use of space-time focusing to obtain images through intervening scattering tissues [3]-[5]. Also, the detectiion of IED (Improvised Explosive Devices) often requires the study of imaging through obscuring layers [2].

Previously, we conducted a theoretical study of space-time imaging of objects which consists of the scattering from the random medium and from a point target [2]. In this paper, we extend our previous study to include a target of finite size and the space-time imaging pattern observed at the receiver.

The pulse incident on the object consists of the coherent and the incoherent components. The coherent component diminishes with the optical depth (OD) while the incoherent component tends to increase with OD. The back scattering from the random medium increases with OD and saturates at large OD. Therefore, the image from the target tends to decrease with OD, while the image from the medium tends to increase with OD. At small OD, the target image dominates while the medium image is dominant at large OD. This paper gives a quantitative comparison of the target and the medium image in terms of OD, aperture size, object size, bandwidth, the focusing, and the medium location.

We start with the general formulations and continue with the discussion of two-frequency fourth order moments and two-frequency mutual coherence functions. The scattering from the medium and the scattering from the object are shown with numerical examples. This paper is dedicated to Professor Leo Felsen who made great contributions in many areas including waves in random media, and whom many of us considered our kind and warm mentor and friend.

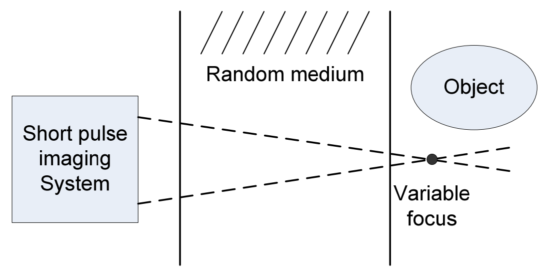

Fig. 1: Imaging system may be an array of short-pulse transmitters and receivers. The received signal is processed to create the image in space and time. It can be a focusing lens system. The object can be inside or behind the random medium.

II. General formulation

We consider an imaging system shown in Fig. 1. The imaging system may be an array of transmitters emitting short pulses and receivers, and the output is processed to obtain the image in space and time. The transmitter and receiver have variable focusing capability, and the received image can be displayed both in space and time. An equivalent imaging system can be a focusing lens and the output image may be viewed both in time and space. In this paper, we use the lens system to explain the basic idea.

Let us consider the lens imaging system shown in Fig. 2. At $z=0$, we have a focusing imaging aperture. A short pulse $f(t)$ with its Fourier transform $F(\omega)$ is emitted at S and propagates to $\bar{p}$ in the aperture. For example, a Gaussian modulated pulse is given by

$$\begin{aligned} f(t) &= A_o \exp\left(-i\omega_o t - \frac{t^2}{T^2}\right) \\ F(\omega) &= A_o \frac{2\sqrt{\pi}}{\Delta\omega} \exp\left(-\frac{(\omega-\omega_o)^2}{\Delta\omega^2}\right) \end{aligned} \tag{1}$$

Manuscript received April 1, 2006. This work was supported by the Office of Naval Research, Code 321, Grant # N00014-15-1-0843 and Grant # N00014-04-1-0074
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where $\Delta\omega = 2/T_o$ is the bandwidth and ω_o is the carrier frequency. The source point S is located at the focal distance f_o from the lens, and thus the field at $\overline{p}$ is given by $G_o(\overline{p})U_o(\overline{p})$ where $G_o(p)$ is the free space Green's function and $U_o(\overline{p})$ is the focusing function. Here we start with a single frequency, and later the two-frequency mutual coherence functions will be used, and the results will be Fourier-transformed to obtain the complete space-time solution. In parabolic approximation, we have

$$G_o(\overline{p}) = \frac{\exp\left(ikf_o + ik\dfrac{p^2}{2f_o}\right)}{4\pi f_o}$$
$$U_o(\overline{p}) = \exp\left(-ik\frac{p^2}{2f_o}\right) \tag{2}$$

This will create a constant phase at the aperture $z = 0$.

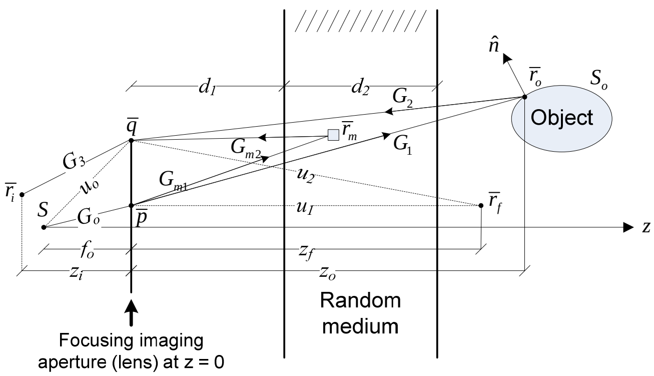

Fig. 2: Imaging system. Focusing aperture is located at $z = 0$. The point source at S emits a short pulse which propagates to $\overline{p}$ on the aperture plane, propagates through the random medium reaching the object, and scattered through the random medium reaching $\overline{q}$. The focusing imaging aperture focuses the pulse on $\overline{r}_f$. This is then focused and forms the image at $\overline{r}_i(z_i, \overline{\rho}_i)$. $\overline{r}_o(z_o, \overline{\rho}_o)$ is the point on the object, $\overline{r}_m(z_m, \overline{\rho}_m)$ is a point in the random medium, and $\overline{r}_i(z_i, \overline{\rho}_i)$ is the image point. The pulse from $\overline{p}$ propagates into the medium at $\overline{r}_m$ and is scattered back to $\overline{q}$.

Next, we consider the pulse propagation from $\overline{p}$ to $\overline{r}_o$ on the object and back to $\overline{q}$. We assume that the object has Dirichlet surface and the radius of curvature of the surface is large and the Kirchhoff approximation is valid [6]. Thus, we get the field at $\overline{q}$ given by

$$-2\int G_2(\overline{q}, \overline{r}_o)\frac{\partial}{\partial n}G_1(\overline{r}_o, \overline{p})G_o(\overline{p})u_o(\overline{p})F(\omega)\, dS_o \tag{3}$$

where G_1 and G_2 are the stochastic Green's function from $\overline{p}$ to $\overline{r}_o$ and from $\overline{r}_o$ to $\overline{q}$, respectively. We also note that within the parabolic approximation, G_1 is traveling mostly in the z direction, and therefore, we have approximately

$$\frac{\partial}{\partial n} = ik\hat{z}\cdot\hat{n} \tag{4}$$

Thus, the field at $\overline{q}$ is given by

$$i2k\int_{S_o} G_2(\overline{q}, \overline{r}_o)G_1(\overline{r}_o, \overline{p})G_o(\overline{p})u_o(\overline{p})F(\omega)\, d\overline{\rho}_o \tag{5}$$

where $\overline{r}_o = \overline{r}_o(z_o, \overline{\rho}_o)$ and S_o is the object surface.

Next, we consider the pulse propagation from $\overline{p}$ to a medium point $\overline{r}_m(z_m, \overline{\rho}_m)$ to $\overline{q}$. We have the field at $\overline{q}$ due to a particle located at $\overline{r}_m$ where particles are randomly distributed

$$\psi_m = G_{m2}(\overline{q}, \overline{r}_m)4\pi f\, G_{m1}(\overline{r}_m, \overline{p})G_o(\overline{p})u_o(\overline{p})F(\omega) \tag{6}$$

where f is the scattering amplitude of the particle. Here G_{m2} and G_{m1} are the stochastic Green's functions.

Let us next consider the focusing function which focuses the field on $\overline{r}_f(z_f, \overline{\rho}_f)$ with Gaussian amplitude distribution $\exp(-p^2/a^2)$ at $z = 0$ where a is the aperture size. The Green's function G_1 is then modified by the focusing function u_1

$$u_1 = \exp\left(-ik\frac{\left|\overline{p} - \overline{\rho}_f\right|^2}{2Z_f} - \frac{p^2}{a^2}\right) \tag{7}$$

Furthermore, the wave received at $\overline{q}$ is also modified by the focusing function u_2

$$u_2 = \exp\left(-ik\frac{\left|\overline{q} - \overline{\rho}_f\right|^2}{2Z_f} - \frac{q^2}{a^2}\right) \tag{8}$$

The field at $\overline{q}$ then propagates with free space Green's function G_3 with focusing function u_3 and reaches the imaging point $\overline{r}_i(z_i, \overline{\rho}_i)$

$$u_3 = \exp\left(-\frac{ikq^2}{2f_o}\right)$$
$$G_3 = \frac{1}{4\pi z_i}\exp\left(ikz_i + ik\frac{|\overline{q} - \overline{\rho}_i|^2}{2z_i}\right) \tag{9}$$

Summarizing the above, we get the field at image point $\overline{r}_i$ due to the scattering by the object

$$\psi(\overline{r}_i) = \int (i2k)\, d\overline{\rho}_o \psi_o$$
$$\psi_o = \iint d\overline{p}\; d\overline{q}\; G_3u_3\; G_2u_2\; G_1u_1\; G_ou_o\; F(\omega) \tag{10}$$

The field due to a particle in the medium is given by $\psi_m(4\pi f)$ where

$$\psi_m = \iint d\overline{p}\; d\overline{q}\; G_3u_3\; G_{m2}u_2\; G_{m1}u_1\; G_ou_o\; F(\omega) \tag{11}$$

We now consider the temporal behavior of the wave at the imaging point. We have

$$\psi(\overline{r}_i, t) = \frac{1}{2\pi}\int d\omega\psi(\overline{r}_i)\exp(-i\omega t) \tag{12}$$

$$\psi(\overline{r}_i) = \psi(\overline{r}_i, \omega)$$
$$= \int_{S_o} i2k\psi_o(\omega)\, d\overline{\rho}_o + \sum_{\mathrm{m}}\psi_{\mathrm{m}} \tag{13}$$

Where ψ_o is given in (10) and ψ_m is given in (11). The summation over m is the contributions from all particles.

We now consider the temporal intensity at $\overline{r}_i$.

$$I\left(\overline{r}_i,t\right) = \frac{1}{\left(2\pi\right)^2}\int d\omega\int d\omega'\left\langle\psi\left(\overline{r}_i,\omega\right)\psi^*\left(\overline{r}_i,\omega'\right)\right\rangle\exp\left(-i(\omega-\omega')t\right) \quad (14)$$

The two-frequency mutual coherence function Γ is then the sum of the object scattering Γ_o and the medium scattering Γ_m.

$$\begin{aligned}\Gamma\left(\omega,\omega'\right) &= \left\langle\psi\left(\overline{r}_i,\omega\right)\psi^*\left(\overline{r}_i,\omega'\right)\right\rangle \\ &= \Gamma_o\left(\omega,\omega'\right)+\Gamma_m\left(\omega,\omega'\right) \\ \Gamma_o\left(\omega,\omega'\right) &= \int_{S_o} d\overline{\rho}_o\int_{S_o} d\overline{\rho}_o' 4k_1k_2\left\langle\psi_o\left(\omega\right)\psi_o^*\left(\omega'\right)\right\rangle \\ \Gamma_m\left(\omega,\omega'\right) &= \int_V\left\langle\psi_m\left(\omega\right)\psi_m^*\left(\omega'\right)\right\rangle\left(4\pi\right)^2\sigma_d\rho_n\ dV\end{aligned} \quad (15)$$

where $\sigma_d = f\left(\omega\right)f^*\left(\omega'\right)$ is the two-frequency differential cross section of a single particle, ρ_n is the number density, V is the volume of the random medium, and S_o is the surface of the object.

Equation (14), together with (15) constitute the final expression of the intensity as a function of time and the image point $\overline{r}_i$. In deriving this, we made several assumptions. We assumed the wave propagation is represented by the parabolic approximation, the object surface is smooth so that the Kirchhoff approximation is valid, the scattering from the object and the scattering from the random medium are uncorrelated, and the random medium is represented by a random distribution of particles. If the random medium is a random continuum, such as turbulence, we can use the above formulation by using the turbulence spectrum in place of the phase function of particles.

III. Image due to Scattering from the Object and from the Medium

Let us first consider the temporal intensity of the scattered wave from the object at the image point $\overline{r}_i$ given in (14) and (15).

$$I_o(\overline{r}_i,t) = \frac{1}{\left(2\pi\right)^2}\int d\omega\int d\omega'\Gamma_o\left(\omega,\omega'\right)\exp\left(-i(\omega-\omega')t\right) \quad (16)$$

In order to calculate Γ_o in (15), we need to evaluate $\left\langle\psi_o\left(\omega\right)\psi_o^*\left(\omega'\right)\right\rangle$ given in (10).

Let us write

$$\psi_o(\omega) = \iint d\overline{p}\ d\overline{q}\ M(\omega)G_2\left(\omega\right)G_1\left(\omega\right) \quad (17)$$

where $M(\omega) = G_3u_3u_2u_1G_ou_oF$ is deterministic and G_2 and G_1 are the stochastic Green's function. We then get

$$\begin{aligned}\left\langle\psi\left(\overline{r}_i,\omega\right)\psi^*\left(\overline{r}_i,\omega\right)\right\rangle &= \iint d\overline{p}\ d\overline{q}\ d\overline{p}'\ d\overline{q}'\ M(\omega)M(\omega') \\ &\quad \left\langle G_2(\omega)G_1(\omega)G_2^*(\omega')G_1^*(\omega')\right\rangle\end{aligned} \quad (18)$$

Here, we need to evaluate the fourth order moment. This can be expressed in terms of the second moments using the circular complex Gaussian assumption. We get

$$\begin{aligned}\left\langle G_2G_1G_2^*G_1^*\right\rangle &= \left\langle G_2G_2^*\right\rangle\left\langle G_1G_1^*\right\rangle+\left\langle G_2G_1^*\right\rangle\left\langle G_1G_2^*\right\rangle \\ &\quad -\left\langle G_2\right\rangle\left\langle G_1\right\rangle\left\langle G_2^*\right\rangle\left\langle G_1^*\right\rangle\end{aligned} \quad (19)$$

The coherent field in (19) is given by

$$\begin{aligned}\left\langle G_1\right\rangle &= \frac{1}{4\pi z_o}\exp\left[ik\left(z_o+\frac{\left|\overline{\rho}_o-\overline{p}\right|^2}{2z_o}\right)-\frac{\tau_o}{2}\right] \\ \left\langle G_2\right\rangle &= \frac{1}{4\pi z_o}\exp\left[ik\left(z_o+\frac{\left|\overline{\rho}_o-\overline{q}\right|^2}{2z_o}\right)-\frac{\tau_o}{2}\right] \\ \left\langle G_1^*\right\rangle &= \frac{1}{4\pi z_o'}\exp\left[-ik'\left(z_o'+\frac{\left|\overline{\rho}_o'-\overline{p}'\right|^2}{2z_o'}\right)-\frac{\tau_o}{2}\right] \\ \left\langle G_2^*\right\rangle &= \frac{1}{4\pi z_o'}\exp\left[-ik'\left(z_o'+\frac{\left|\overline{\rho}_o'-\overline{q}'\right|^2}{2z_o'}\right)-\frac{\tau_o}{2}\right]\end{aligned} \quad (20)$$

where τ_o is the optical depth, $k = \omega/c$ and $k' = \omega'/c$, and the point on the object is $\overline{r}_o\left(z_o,\overline{\rho}_o\right)$ and $\overline{r}_o'\left(z_o',\overline{\rho}_o'\right)$ shown in Fig. 3.

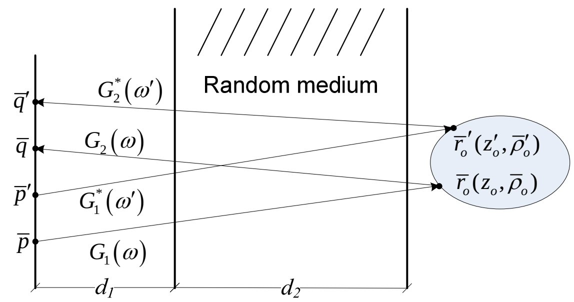

Fig. 3: Calculation of the two-frequency fourth order moments.

The two-frequency mutual coherence function in (19) can be expressed as follows: [2][7]

$$\left\langle G_iG_j^*\right\rangle = \left\langle G_{oi}\right\rangle\left\langle G_{oj}^*\right\rangle\exp\left(-H_{ij}\right) \quad (21)$$

where H_{ij} is given by

$$H_{ij} = \tau_a+\int_0^L dz\frac{b}{2}\int_0^\infty s\ ds\left[1-gJ_o\left(k_csP_{ij}\right)\right]p(s) \quad (22)$$

G_{oi} and G_{oj} are the free space Green's functions for G_i and G_j, a_b is the absorption coefficient, b is the scattering coefficient, $\tau_a = \int a_b\ dz$ is the absorption depth, $p(s)$ is the phase function, $s = 2\sin\left(\theta/2\right)$, θ is the scattering angle, and g is the two-frequency factor given by

$$g = \exp\left[i\frac{z\left(z_l-z\right)}{z_l}k_ds^2\right],\quad k_d = k_1-k_2 \quad (23)$$

$k_c = \left(k_1+k_2\right)/2$ is the center frequency, $z_l = \left(z_o+z_o'\right)/2$ is the average distance to the object. The factor P_{ij} is given by

$$
\begin{aligned}
P_{11} &= \left| \left(\bar{p}-\bar{p}'\right)\left(1-\frac{z}{z_l}\right)+\left(\bar{\rho}_o-\bar{\rho}_o'\right)\frac{z}{z_l}\right| \\
P_{22} &= \left| \left(\bar{q}-\bar{q}'\right)\left(1-\frac{z}{z_l}\right)+\left(\bar{\rho}_o-\bar{\rho}_o'\right)\frac{z}{z_l}\right| \\
P_{12} &= \left| \left(\bar{p}-\bar{q}'\right)\left(1-\frac{z}{z_l}\right)+\left(\bar{\rho}_o-\bar{\rho}_o'\right)\frac{z}{z_l}\right| \\
P_{21} &= \left| \left(\bar{q}-\bar{p}'\right)\left(1-\frac{z}{z_l}\right)+\left(\bar{\rho}_o-\bar{\rho}_o'\right)\frac{z}{z_l}\right|
\end{aligned}
\tag{24}
$$

Generalized two-frequency mutual coherence function is summarized in Appendix A. Using (21) and (22), we write (19):

$$
\begin{aligned}
\left\langle G_2 G_1 G_2^* G_1^* \right\rangle &= G_{2o} G_{1o} G_{2o}^* G_{1o}^* \\
&\left[\exp\left(-H_{22}-H_{11}\right)+\exp\left(-H_{21}-H_{12}\right)-\exp(-2\tau_o)\right]
\end{aligned}
\tag{25}
$$

The image due to the scattering from the medium requires evaluation of $\left\langle \psi_m(\omega)\psi_m^*(\omega')\right\rangle$ in (15). This is given by the same expression as (18), except that $\bar{r}_o=\bar{r}_o'=\bar{r}_m$, and thus $\bar{\rho}_o=\bar{\rho}_o'=\bar{\rho}_m$.

We now summarize the final expression of the intensity $I(\bar{r}_i,t)$

At the image point $\bar{r}_i$ at time t.

$$
I(\bar{r}_i,t)=I_o(\bar{r}_i,t)+I_m(\bar{r}_i,t) \tag{26}
$$

where I_o is the contribution from the object and I_m is the contribution from the medium. We have

$$
\begin{aligned}
I_o(\bar{r}_i,t) &= \frac{1}{(2\pi)^2}\int d\omega \int d\omega' \Gamma_o(\omega,\omega')\exp\left(-i\left(\omega-\omega'\right)t\right) \\
\Gamma_o(\omega,\omega') &= \int_{S_o} d\bar{\rho}_o \int_{S_o} d\bar{\rho}_o' \, 4kk' \int d\bar{p}\int d\bar{q}\int d\bar{p}'\int d\bar{q}' \\
& M(\omega)M^*(\omega)G_{1o}G_{2o}G_{1o}^*G_{2o}^* \\
& \left[\exp\left(-H_{22}-H_{11}\right)+\exp\left(-H_{21}-H_{12}\right)-\exp(-2\tau_o)\right]
\end{aligned}
\tag{27}
$$

where G_{1o}, G_{2o} are the free space Green's functions.

The medium contribution is given by

$$
\begin{aligned}
I_m(\bar{r}_i,t) &= \frac{1}{(2\pi)^2}\int d\omega \int d\omega' \Gamma_m(\omega,\omega')\exp\left(-i\left(\omega-\omega'\right)t\right) \\
\Gamma_m(\omega,\omega') &= \int_V dV \left(4\pi\right)^2 \sigma_d \rho_n \int d\bar{p}\int d\bar{q}\int d\bar{p}'\int d\bar{q}' \\
& M(\omega)M^*(\omega)G_{1o}G_{2o}G_{1o}^*G_{2o}^* \\
& \left[\exp\left(-H_{22o}-H_{11o}\right)+\exp\left(-H_{12o}+H_{21o}\right)-\exp(-2\tau_o)\right]
\end{aligned}
\tag{28}
$$

where

$$
H_{ijo}=H_{ij} \text{ with } \bar{\rho}_o=\bar{\rho}_o' \tag{29}
$$

Equation (26) with (27), (28) and (29) are the final expression for the intensity in space and time at the image point.

IV. Evalution of Two-Frequency Mutual Coherence Function for Gaussian Phase Function

Even though (27) and (28) give the complete solution for our problem, they involve multiple integrals which need to be evaluated. If we assume Gaussian phase function, we can simplify (27) and (28) and analytically perform the integration with respect to $d\bar{p}d\bar{p}'d\bar{q}d\bar{q}'$.

First we note that H_{ij} in (22) can be expressed with the Gaussian phase function:

$$
p(s)=4\alpha_p \exp\left(-\alpha_p s^2\right) \tag{30}
$$

We get

$$
H_{ij}=\tau_a+\int dz b\left[1-\frac{\exp\left(-\frac{k^2 P_{ij}}{4\left(\alpha_p-A\right)}\right)}{1-\frac{A}{\alpha_p}}\right] \tag{31}
$$

where $A=\frac{iz\left(z_l-z\right)}{2z_l}k_d$, $k_d=k-k'$, $z_l=\frac{z_o+z_o'}{2}$. This can be further approximated and we get [2]

$$
\begin{aligned}
\exp\left(-H_{ij}\right) &= \exp\left(-\tau_o\right)+F_s X_{ij} \\
X_{ij} &= \exp\left(-\tau_a\right)+\int dz \; b\left[\left(1-B\right)+\frac{Bk^2P_{ij}^2}{F_s 4\left(\alpha_p-A\right)}\right]
\end{aligned}
\tag{32}
$$

where $B=\left(1-A/\alpha_p\right)^{-1}$, $F_s=1-\exp\left(-\int dz \; bB\right)$. P_{ij} is given in (24). We now get

$$
\begin{aligned}
&\exp\left(-H_{22}-H_{11}\right)+\exp\left(-H_{21}-H_{12}\right)-\exp\left(-2\tau_o\right)= \\
&\quad \exp\left(-2\tau_o\right)+F_s\exp\left(-\tau_o\right)\left(X_{11}+X_{22}+X_{12}+X_{21}\right) \\
&\quad +F_s^2\left(X_{11}X_{22}+X_{12}X_{21}\right)
\end{aligned}
\tag{33}
$$

Here the three terms with X_{12} and X_{21} represent the coherent backscattering. Substituting (33) into (27), we can analytically perform the integration with respect to $d\bar{p}d\bar{p}'d\bar{q}d\bar{q}'$, with the repeated use of the formula.

$$
\int d\bar{p}\exp\left(-Ap^2+\bar{B}\cdot\bar{p}\right)=\frac{\pi}{A}\exp\frac{\left(\bar{B}\cdot\bar{B}\right)}{4A}, \quad \operatorname{Re}\left\{A\right\}>0 \tag{34}
$$

Similar calculations can be made for the medium scattering (28), with $\bar{\rho}_o=\bar{\rho}_o'$. The backscattering coefficient $\sigma_d\rho_n$ can be expressed as $bp\left(s=2\right)$. $s=2\sin\left(\theta/2\right)=2$ at $\theta=\pi$. Here $p\left(s=2\right)$ using (30) gives a poor approximation for backscattering. We should use a more realistic value using Henyey-Greenstein phase function. We get

$$
bp\left(s=2\right)=\frac{1-g_a}{\left(1+g_a\right)^2}b \tag{35}
$$

where g_a = anisotropy factor.

For a given anisotropy factor g_a, α_p in (30) is determined by equating the same half-power beamwidth for Henyey-Greenstein and the Gaussian phase function. We have

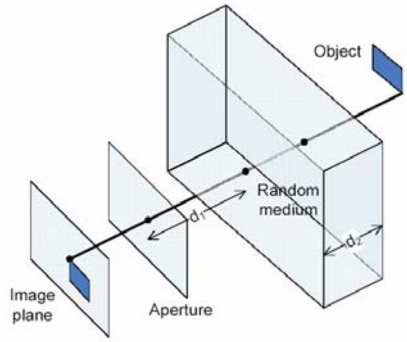

Fig. 4: Example of imaging through a random medium.

$$\alpha_p = \frac{(\ln 2) g_a}{(1-g_a)^2 (2^{2/3}-1)} \tag{36}$$

For $g_a = 0.85$, we get $\alpha_p = 44.6$.

Equation (16) gives a complete space-time solution applicable to narrow as well as wide band cases. However, if the bandwidth is small, we can approximate (16)

$$\begin{aligned} &\frac{1}{(2\pi)^2}\int d\omega \int d\omega' F(\omega) F(\omega') \Gamma(\omega,\omega') \exp(-i(\omega-\omega')t) \\ &\approx \frac{1}{\pi}\frac{1}{\Delta\omega}\sqrt{\frac{\pi}{2}}\int d\omega_d \exp(-i\omega_d t)\Gamma(\omega_d)\exp\left(-\frac{\omega_d^2}{2\Delta\omega^2}\right) \end{aligned} \tag{37}$$

V. Numerical Examples

As an example, we consider the imaging of objects shown in Fig. 4. We used $z_o = 50\lambda$, $z_i = 5\lambda$, $a = 5\lambda$, $d_1 = 25\lambda$, $d_2 = 25\lambda$, $\alpha_p = 44.58$, $W_o = 0.9$ (albedo), the object size is $5\text{cm}\times 5\text{cm}$, $\lambda = 3$ cm, scanning image area at $z_i = 2\text{cm}\times 2\text{cm}$, and number of frequency points is 10. This is used as the reference case. The time is evaluated at $2(f_o + z_o)/c$. We choose the imaging plane at $z_i = f_o$. Also we put the object at the focal plane $z_o = z_f$, and the focal point on axis $\bar{\rho}_f = 0$. This is a special example. Our formula, however, is applicable to more general case shown in Fig. 2. Note that image size is ideally $5\text{mm}\times 5\text{mm}$ square in the third quadrant because of the ratio $z_i / z_o = 0.1$. This will be blurred due to the finite aperture size and the random medium.

We now consider several cases.

(a) Object size:

Fig. 5 (a) shows the image pattern at OD = 1, which consists of the target image and the medium image. Note that the target image is located in the image box, but the medium image is centered at the origin as expected. The relative peak values of the target and the medium image are shown in Fig. 5 (b). This also shows a $5\text{cm}\times 5\text{cm}$ target (reference) and $10\text{cm}\times 10\text{cm}$ target. Note that the medium image is the same, but the target image is about 6 dB higher for the $10\text{cm}\times 10\text{cm}$ target than the $5\text{cm}\times 5\text{cm}$ target. This shows that the wave at the target is mostly incoherent, and therefore, the target image is in proportion to the area.

Figures 5 (c) and (d) show the image pattern for a $5\text{cm}\times 5\text{cm}$ object and $10\text{cm}\times 10\text{cm}$ object at OD=8. As seen from Fig. 5 (b), the target image is higher for a $10\text{cm}\times 10\text{cm}$ object than for a $5\text{cm}\times 5\text{cm}$ object.

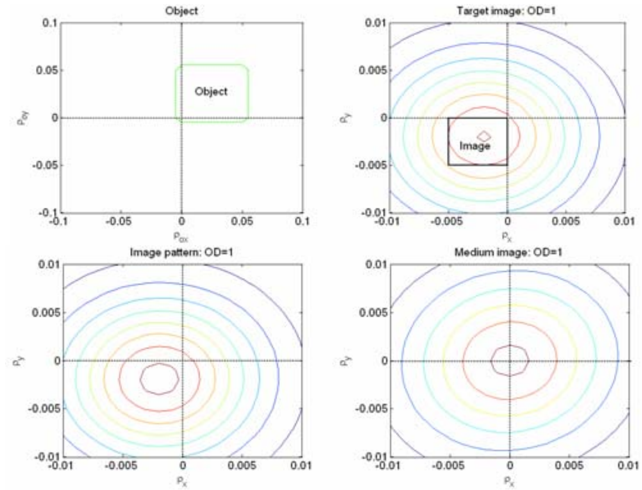

Fig. 5(a): Image pattern, target image and medium at OD = 1.

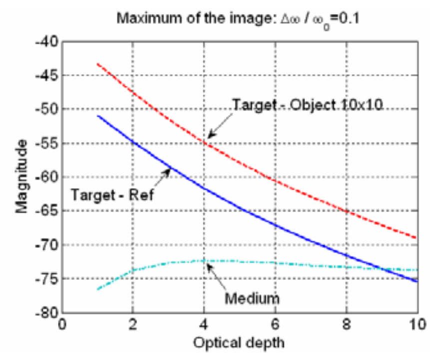

Fig. 5(b): Relative peak values of the target and medium images.

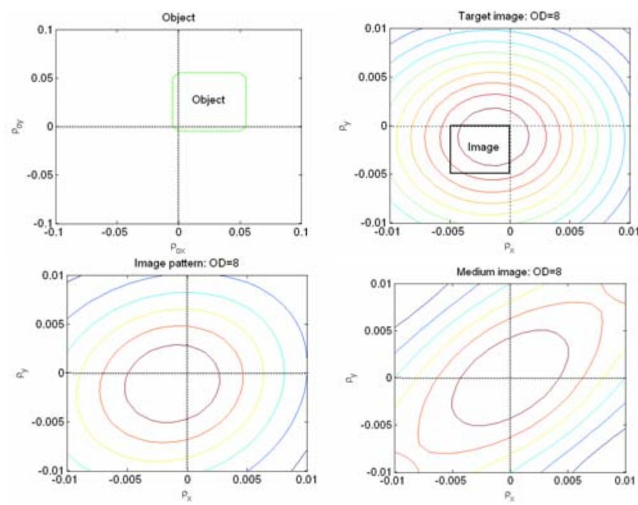

Fig. 5(c): Image pattern, target image and medium at OD = 8 for a $5\text{cm}\times 5\text{cm}$ object.

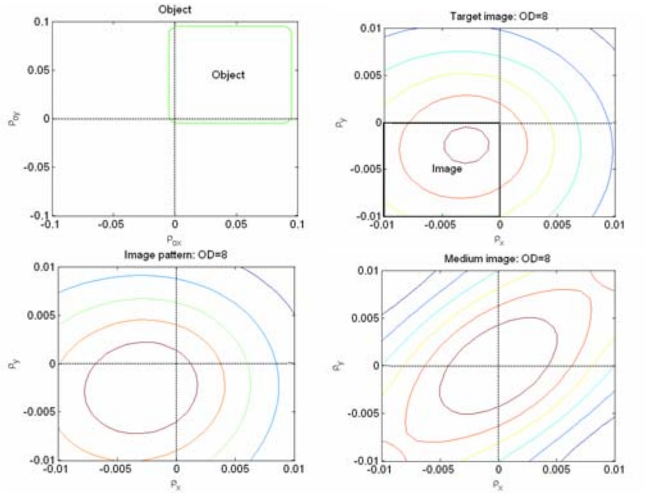

Fig. 5(d): Image pattern, target image and medium at OD = 8 for a $10\text{cm} \times 10\text{cm}$ object.

(b) Aperture size:
Fig. 6 shows the peak values of the target image and the medium image showing increased target and medium contributions for a larger aperture size.

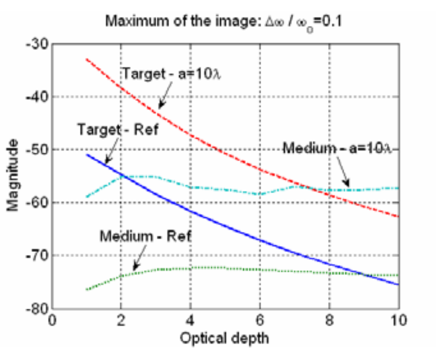

Fig. 6: Aperture size $a = 5\lambda$ and $a = 10\lambda$.

(c) Location of the random medium (shower curtain effect)
Fig. 7 shows the effects of the location of the random medium, commonly called the "shower curtain effect". Note that when the medium is close to the object $d_1 = 25\lambda$, both target and medium contributions are larger than where the medium is away from the object ($d_1 = 12\lambda$).

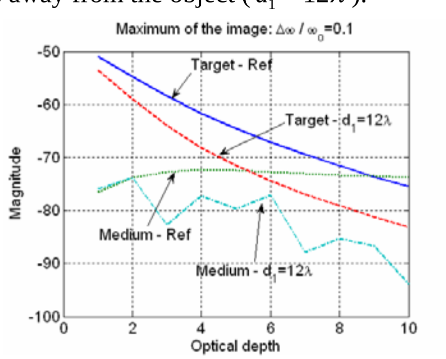

Fig. 7: Shower curtain effects.

(d) Bandwidth

Reduction of the bandwidth broadens the pulse width. Fig. 9 shows that the pulse return from the target as a function of time is broader for BW=0.001 than for BW=0.1. Note that for BW=0.01, it is impossible to distinguish target and medium scattering, but for BW=0.1, it is clearly possible to distinguish them.

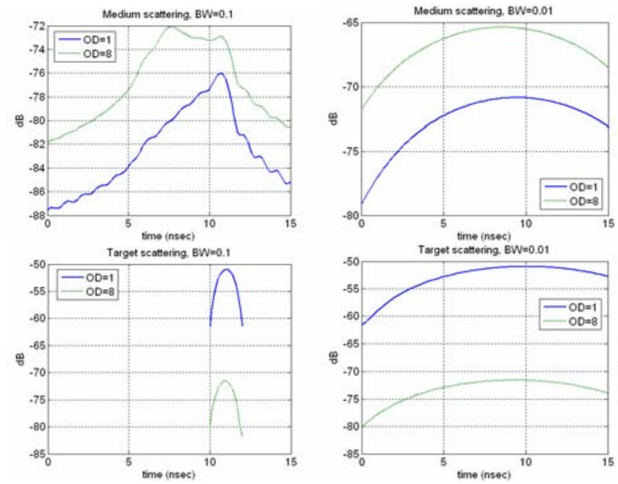

Fig. 9: Effects of different bandwidths.

VI. Conclusions

We have presented a theory of imaging of objects through a random medium. A short focused pulse is emitted at the aperture and the received signal is then displayed as a spatial image at different times. Our formulation and computer code are general, including the object located at an arbitrary point, and the focal point can be at any point. The imaging system includes the imaging point at any point at an arbitrary time, even though our numerical examples are for the image plane at the same source point and for time given by the exact time of travel from the source to the object and back to the image plane. The focal point can be varied or scanned. The imaging system should be useful as a device to probe an object hidden behind random obscuring layers. The theory presented in this paper includes the study of the two-frequency fourth order moments, which have received little attention in the past, but may be useful in other applications for space-time propagation and scattering in random media.

Appendix A: Generalized Two-Frequency Mutual Coherence Function

A general case involves two separate correlated sources as shown in Fig. A-1.
We assume that $|z_1 - z_2| << L_e$, $|z_{10} - z_{20}| << L_e$, where L_e is the equivalent average distance

$$L_e = \frac{(z_1 + z_2)}{2} - \frac{(z_{10} + z_{20})}{2} \quad \text{(A-1)}$$

We get the two-frequency mutual coherence function

$$\Gamma(\bar{r}_1, \omega_1; \bar{r}_2, \omega_2) = G_{10}(\bar{r}_1, \omega_1) G_{20}^*(\bar{r}_2, \omega_2) \exp(-H)$$

$$H = 4\pi^2 \int_0^L dz \int_0^\infty \kappa d\kappa \left[\frac{k_1^2 + k_2^2}{2} - k_1 k_2 g J_o(\kappa P) \right] \Phi_n(\kappa, z) \quad \text{(A-2)}$$

$\Phi_n(\kappa, z)$ is the spectrum of the random continuum. For particles, we use the phase function $p(s)$.

$$2\pi k^4 \Phi_n(\kappa) = \frac{b}{4\pi} p(s), \quad \kappa = ks .$$

We have under parabolic approximation

$$\begin{aligned}
G_{10} &= \frac{1}{4\pi|z_1 - z_{10}|} \exp\left[ik_1 |z_1 - z_{10}| + i \frac{k_1 |\bar{\rho}_1 - \bar{\rho}_{10}|^2}{2|z_1 - z_{10}|} \right] \\
G_{20}^* &= \frac{1}{4\pi|z_2 - z_{20}|} \exp\left[ik_2 |z_2 - z_{20}| + i \frac{k_2 |\bar{\rho}_2 - \bar{\rho}_{20}|^2}{2|z_2 - z_{20}|} \right] \\
g &= \exp\left[-i \frac{z(L_e - z)}{2L_e} \left(\frac{1}{k_1} - \frac{1}{k_2} \right) \kappa^2 \right] \\
P &= \left| (\bar{\rho}_{10} - \bar{\rho}_{20}) \left(1 - \frac{z}{L_e} \right) + (\bar{\rho}_1 - \bar{\rho}_2) \frac{z}{L_e} \right| .
\end{aligned} \quad \text{(A-3)}$$

VII. REFERENCES

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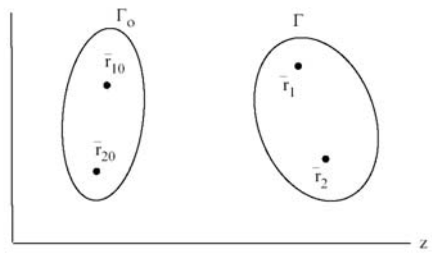

Fig. A-1: Two-frequency MCF at $\bar{r}_1(z_1, \bar{\rho}_1)$ and $\bar{r}_2(z_2, \bar{\rho}_2)$ duet to two-frequency MCF at $\bar{r}_{10}(z_{10}, \bar{\rho}_{10})$ and $\bar{r}_{20}(z_{20}, \bar{\rho}_{20})$.

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