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Second-order scattering approximation of pulse vector radiative transfer equation

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Abstract

The problem of polarized light propagating through scattering media can be explained using the vector radiative transfer equation. This equation is an integro-differential equation and is well-known to be unsolvable analytically. One of the approximate solutions is discrete ordinates method which is based on the discretization of the Stokes parameters and the Mueller matrix. Although it produces accurate results, it requires a lot of computational resources. In addition, there are limitations on the calculation for angles that are very close to the optical axis. The solutions at these angles are necessary for some applications such as atmospheric imaging. First-order scattering approximation has been applied to mitigate the computational resource situation. It can also be used to calculate the solution at the angles that are very close to optical axis. However, it lacks information about the cross-polarization and it is inaccurate when light encounters more scattering events. Second-order scattering approximation provides more accurate solutions and offers some information about cross-polarization. We develop the first-order and second-order scattering approximations and their solutions for the pulse wave case. We investigate the second-order approximation solutions and compare them to the solution from the complete vector radiative transfer equation in several cases.

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1. Introduction

The behaviors of polarized light propagating through random media have recently been under investigation. The findings will be valuable to many practical problems such as imaging through atmosphere, biomedical imaging, etc. It is well known that the Vector Radiative Transfer Equation (VRTE) is an integro-differential equation. As of now, there is no complete analytical solution to this equation. However,

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there are several effective approximations. The Discrete Ordinates Method (DOM) is one of the standard methods [1–4]. It is based on the angular discretization of the Stokes parameter and the Mueller matrix. Then, the integro-differential equation can be transformed into a much simpler matrix equation. The main disadvantage of this method is the substantial amount of computational resource required for the solutions. The matrix equation solver requires a lot of memory and computational time, especially in the case of very small time and angle resolutions.

First-order scattering approximation is much simpler and applicable in the cases where the scattering effect is small [4]. It can be employed to calculate for small time and angle resolution. However, it does not contain information about the cross-polarization. Second-order scattering approximation provides more accurate solutions [6]. It also provides information about the cross-polarization. Thus, second-order scattering approximation is a much easier way to solve for very small time and angle resolutions while providing rather complete description of the co-polarized and cross-polarized components. The purpose of this chapter is to establish the valid range where the first-order and second-order approximations are accurate and to compare the solution of first-order and second-order approximations to the solution of the complete radiative transfer equation.

In Section 2, we explain the nature of the problem and derive the first-order and second-order scattering approximations from the pulse vector radiative transfer equation. Section 3 expresses the solution in the linear and circular polarization cases under certain assumptions. In Section 4, we numerically calculate the solution in several cases. The comparison between radiative transfer and the first-order and second-order approximation are explained.

2. First-order and second-order approximations of the vector radiative transfer equation

We consider polarized light propagating through a slab of random medium in a plane parallel geometry shown in Fig. 1. The random medium, in this case, is randomly located dielectric spheres suspended in a homogeneous background. Such media can be realized in practical environments such as fog or clouds in the air, biological tissues, etc. The optical distance τ is defined by $\tau = \rho\sigma_t z$, where ρ is the number density, σ_t is the total cross-section of a single particle, and z is the actual distance in the medium. The optical depth τ_0 is defined by $\rho\sigma_t L$, where L is the thickness of the slab of random medium.

The equation that governs the behaviors of the light pulse propagating through a source free discrete random medium is the pulse Vector Radiative Transfer Equation (VRTE). The frequency dependent form of this equation is expressed in Eq. (1)

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}(\omega,\tau,\mu,\phi) + \left(1 + (\mu-1)\mathrm{i}\frac{\omega}{\tau_0}\right)\mathbf{I}(\omega,\tau,\mu,\phi)$$
$$= \int_0^{2\pi}\int_{-1}^{1}\mathbf{S}(\mu,\phi,\mu',\phi')\mathbf{I}(\omega,\tau,\mu',\phi')\,\mathrm{d}\mu'\,\mathrm{d}\phi' \quad \text{for } 0 \leqslant \tau \leqslant \tau_0, \tag{1}$$

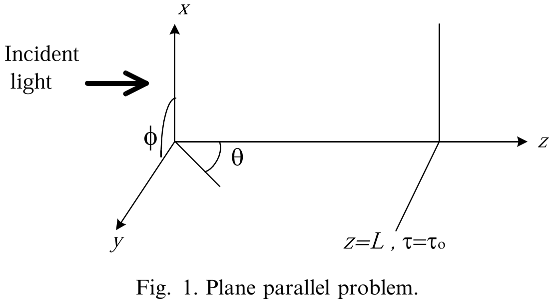

Fig. 1. Plane parallel problem.

where $\mu = \cos\theta$ is the cosine of the polar angle. This equation can be solved with the boundary conditions of

$$\mathbf{I}(\tau = 0) = 0 \quad \text{for } 0 \leqslant \mu \leqslant 1, \tag{2a}$$

$$\mathbf{I}(\tau = \tau_0) = 0 \quad \text{for } -1 \leqslant \mu \leqslant 1, \tag{2b}$$

where $\mathbf{I}$ is the modified Stokes parameter, defined by

$$\mathbf{I} = [I_1 \quad I_2 \quad U \quad V]^{\mathrm{T}}. \tag{3}$$

$\mathbf{S}$ is the Mueller matrix expressed in Eq. (4),

$$\mathbf{S}(\mu,\phi,\mu',\phi') = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix} = \sigma_{\mathrm{t}}^{-1} \begin{bmatrix} |f_{11}|^2 & |f_{12}|^2 & \Re(f_{11}f_{12}^*) & -\Im(f_{11}f_{12}^*) \\ |f_{21}|^2 & |f_{22}|^2 & \Re(f_{21}f_{22}^*) & -\Im(f_{21}f_{22}^*) \\ 2\Re(f_{11}f_{21}^*) & 2\Re(f_{12}f_{22}^*) & 2\Re(f_{11}f_{22}^* + f_{12}f_{21}^*) & -\Im(f_{11}f_{22}^* - f_{12}f_{21}^*) \\ 2\Im(f_{11}f_{21}^*) & 2\Im(f_{12}f_{22}^*) & \Im(f_{11}f_{22}^* + f_{12}f_{21}^*) & \Re(f_{11}f_{22}^* - f_{12}f_{21}^*) \end{bmatrix}, \tag{4}$$

where $f_{11}, f_{12}, f_{21}, f_{22}$ are the scattering amplitudes explained in Appendix A. The submatrices $\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3$, and $\mathbf{S}_4$ are introduced for convenience in the derivation later in this paper. To find the time-dependent parameters, we apply the Fourier transform to Eq. (1). Then, we have

$$\mathbf{I}(t,\tau,\mu,\phi) = \frac{1}{2\pi}\int \mathbf{I}(\omega,\tau,\mu,\phi)\exp\left(\mathrm{i}\frac{\omega}{\tau_0}\tau - \mathrm{i}\omega t\right)\mathrm{d}\omega, \tag{5}$$

where $\mathbf{I}(t,\tau,\mu,\phi)$ is the time-dependent Stokes vector. With the application of the boundary conditions, the solution consists of the reduced and the diffuse Stokes parameters shown below

$$\mathbf{I}(\omega,\tau,\mu,\phi) = \mathbf{I}_{\mathrm{ri}}(\omega,\tau,\mu,\phi) + \mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu,\phi), \tag{6}$$

where the reduced Stokes parameter $\mathbf{I}_{\mathrm{ri}}$ is given y

$$\mathbf{I}_{\mathrm{ri}}(\omega,\tau,\mu,\phi) = \mathbf{I}_0(\omega)\exp(-\tau)\delta(\mu-1)\delta(\phi). \tag{7}$$

Eq. (1) is an integro-differential equation which cannot be solved analytically. One approach to the solution is to transform the equation to a differential equation and apply the boundary conditions. The details of the solution procedures are explained by Cheung and Ishimaru [4] and the solutions for pulse vector cases are explained in our previous paper [1]. In summary, the frequency dependent radiative transfer equation in Eq. (1) is reduced to an ordinary differential equation by the following steps.

(1) Integrate the Mueller matrix with respect to ϕ dependent, that is

$$\mathbf{L}(\mu,\mu') = \int_0^{2\pi} \mathbf{S}(\mu,\mu',\phi'-\phi)\,\mathrm{d}(\phi'-\phi). \tag{8}$$

Then, the VRTE is reduced to

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}(\omega,\tau,\mu) + \left[1 + (\mu-1)\mathrm{i}\frac{\omega}{\tau_0}\right]\mathbf{I}(\omega,\tau,\mu) = \int_{-1}^{1}\mathbf{L}(\mu,\mu')\mathbf{I}(\omega,\tau,\mu')\,\mathrm{d}\mu' \quad \text{for } 0 \leqslant \tau \leqslant \tau_0.$$

For the diffuse component only, the VRTE is given by

$$\mu\frac{\partial}{\partial\tau}\mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu) + \left[1 + (\mu-1)\mathrm{i}\frac{\omega}{\tau_0}\right]\mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu) = \int_{-1}^{1}\mathbf{L}(\mu,\mu')\mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu')\,\mathrm{d}\mu' + \mathbf{F}_0(\mu)\exp(-\tau) \quad \text{for } 0 \leqslant \tau \leqslant \tau_0,$$

where $\mathbf{F}_0 = \mathbf{S}(\mu,1,0)\mathbf{I}_0$.

(2) Apply the Gauss quadrature formula which approximates the integration by summation. The Gauss quadrature formula states that

$$\int_{-1}^{1} f(x)\,\mathrm{d}x \approx \sum_{j=-m}^{m} a_j f(x_j), \tag{9}$$

where the coefficient a_j's are the weighting polynomials [3]. Assume that the function is approximated by Gauss quadrature in the order of N. At a given set of angles of interest $\{\mu_i \,|\, i = -N, \ldots, -1, 1, \ldots, N\}$, the VRTE can be written in the form of

$$\mu_i \frac{\partial}{\partial\tau}\mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu_i) + \left[1 + (\mu_i - 1)\mathrm{i}\frac{\omega}{\tau_0}\right]\mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu_i) = \sum_{j=-N}^{N} \mathbf{L}(\mu_i,\mu_j)\mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu_j) + \mathbf{F}_0(\mu_i)\exp(-\tau) \quad \text{for } 0 \leqslant \tau \leqslant \tau_0.$$

Therefore, we can write the equations for every angle of interest in the form of matrix equation as

$$\frac{\partial}{\partial\tau}\mathbf{I} + \mathbf{A}\mathbf{I} = \mathbf{B}\exp(-\tau), \tag{10}$$

where

$$\mathbf{I} = [\mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu_{-N}) \quad \cdots \quad \mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu_i) \quad \cdots \quad \mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu_N)]^{\mathrm{T}}, \tag{11}$$

$$\mathbf{A}_{i,j} = \frac{1}{\mu_i}\left[1 + (\mu_i - 1)\mathrm{i}\frac{\omega}{\tau_0}\right] - \frac{\mathbf{L}(\mu_i,\mu_j)}{\mu_i} \quad \text{for } i = -N,\ldots,-1,1,\ldots,N \text{ and } j = -N,\ldots,-1,1,\ldots,N, \tag{12}$$

$$\mathbf{B} = \left[\frac{\mathbf{F}_0(\mu_{-N})}{\mu_N} \quad \cdots \quad \frac{\mathbf{F}_0(\mu_i)}{\mu_i} \quad \cdots \quad \frac{\mathbf{F}_0(\mu_N)}{\mu_N}\right]^{\mathrm{T}}. \tag{13}$$

(3) Solve this differential matrix equation numerically. A solution of Eq. (10) may be regarded as a sum of the particular and the complementary solution. The complete solution can be found by applying the boundary conditions given in Eqs. (2a) and (2b).

This method provides accurate solution to the VRTE. However, the solutions are provided only at the angles determined by the Gauss quadrature formula, which depends on the order N. The only way to increase the angular resolution or to investigate the solutions very near the optical axis is to increase N, which requires more computational resources.

Another approach is to treat Eq. (1) as a differential equation form as shown in Eq. (14)

$$\frac{\mathrm{d}y}{\mathrm{d}x} + P(x)y = Q(x), \tag{14}$$

where

$$y = \mathbf{I},\ x = \tau,\ P(x) = v/\mu,\ v = (1 + (\mu - 1)\mathrm{i}(\omega/\tau_0)),$$

and

$$Q(x) = 1/\mu\left(\int\int \mathbf{S}(\mu,\phi,\mu',\phi')\mathbf{I}(\omega,\tau,\mu,\phi)\,\mathrm{d}\mu'\,\mathrm{d}\phi'\right).$$

The solution to Eq. (14) is an integral equation

$$y = c\mathrm{e}^{-\int P(x)\,\mathrm{d}x} + \mathrm{e}^{-\int P(x)\,\mathrm{d}x}\int \mathrm{e}^{\int P(x')\,\mathrm{d}x'} Q(x')\,\mathrm{d}x'. \tag{15}$$

With the application of the boundary conditions, the solution consists of the reduced and the diffuse Stokes parameters as shown in Eq. (6). The reduced Stokes parameter $\mathbf{I}_{\mathrm{ri}}$ is given in Eq. (7) The diffuse Stokes parameter $\mathbf{I}_{\mathrm{d}}$ is expressed as

$$\mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu,\phi)=\int_0^{\tau}\exp\left(-\frac{v}{\mu}(\tau-\tau')\right)\left[\int\int \mathbf{S}(\mu,\phi,\mu',\phi')\mathbf{I}(\omega,\tau,\mu',\phi')\,\mathrm{d}\mu'\,\mathrm{d}\phi'\right]\frac{\mathrm{d}\tau'}{\mu}. \tag{16}$$

Eq. (16) cannot be solved unless some assumptions are made. A way to handle this is to assume that **I** in the integrand is approximately the reduced intensity $\mathbf{I}_{\mathrm{ri}}$ given in Eq. (7). The solution with this assumption is called the first-order scattering approximation. For forward directions $(0<\mu<1)$, it is given by

$$\begin{aligned}\mathbf{I}_{\mathrm{d}}^{(1+)}(\omega,\tau,\mu,\phi)&=\int_0^{\tau}\exp\left(-\frac{v}{\mu}(\tau-\tau')-\frac{v}{\mu_0}\tau'\right)\mathbf{F}(\omega,\mu,\phi)\frac{\mathrm{d}\tau'}{\mu}\\&=\mathbf{F}(\omega,\mu,\phi)\frac{\mu_0}{v(\mu_0-\mu)}\left[\exp\left(-\frac{v}{\mu_0}\tau\right)-\exp\left(-\frac{v}{\mu}\tau\right)\right]\end{aligned} \tag{17}$$

and for backward directions, it is given by

$$\begin{aligned}\mathbf{I}_{\mathrm{d}}^{(1-)}(\omega,\tau,\mu,\phi)&=\int_{\tau}^{\tau_0}\exp\left(-\frac{v}{\mu}(\tau-\tau')-\frac{v}{\mu_0}\tau'\right)\mathbf{F}(\omega,\mu,\phi)\frac{\mathrm{d}\tau'}{(-\mu)}\\&=\mathbf{F}(\omega,\mu,\phi)\frac{\mu_0}{v(\mu_0-\mu)}\left[\exp\left(-\frac{v}{\mu_0}\tau\right)-\exp\left(-\frac{v}{\mu}\tau-v\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)\tau_0\right)\right],\end{aligned} \tag{18}$$

where $\mathbf{F}(\omega,\mu,\phi)=\int\int\mathbf{S}(\mu,\phi,\mu_0,\phi_0)\mathbf{I}_{\mathrm{ri}}(\omega,\tau,\mu',\phi')\,\mathrm{d}\mu'\,\mathrm{d}\phi$, $\mu_0=\cos(\theta_0)$, θ_0 and ϕ_0 indicate the incident angle of the pulse wave.

Notice that the solution of the first-order approximation involves only integration. It does not require a matrix solver. Therefore, it is computationally economical. Also, the solution can be evaluated at any angle μ. Thus, we can investigate angles that are very close to the optical axis. However, the accuracy of our solutions are still limited on the accuracy of the Mueller matrix.

The second-order scattering approximation is derived from the first-order approximation as illustrated in Fig. 2. It consists of the terms from the first-order forward scattering $(\mathbf{I}_{\mathrm{d}}^{(1+)})$ and first-order backward scattering $(\mathbf{I}_{\mathrm{d}}^{(1-)})$. In the forward direction, the second-order approximation is given by

$$\mathbf{I}_{\mathrm{d}}^{(2f)}(\omega,\tau,\mu,\phi)=\mathbf{I}_{\mathrm{d}}^{(2f+)}(\omega,\tau,\mu,\phi)+\mathbf{I}_{\mathrm{d}}^{(2f-)}(\omega,\tau,\mu,\phi), \tag{19}$$

where $\mathbf{I}_{\mathrm{d}}^{(2f+)}$ is due to $\mathbf{I}_{\mathrm{d}}^{(1+)}$ and $\mathbf{I}_{\mathrm{d}}^{(2f-)}$ is due to $\mathbf{I}_{\mathrm{d}}^{(1-)}$

$$\mathbf{I}_{\mathrm{d}}^{(2f+)}(\omega,\tau,\mu,\phi)=\int\int_{0<\mu'<1}\mathbf{S}(\mu,\phi,\mu',\phi')\mathbf{I}_{\mathrm{d}}^{(2fa)}(\omega,\tau,\mu,\mu',\phi)\,\mathrm{d}\phi'\,\mathrm{d}\mu', \tag{20}$$

$$\begin{aligned}\mathbf{I}_{\mathrm{d}}^{(2fa)}(\omega,\tau,\mu,\mu'\phi)&=\int_0^{\tau}\exp\left(-\frac{v}{\mu}(\tau-\tau')\right)\mathbf{I}^{(1+)}(\omega,\tau,\mu,\mu',\phi)\,\frac{\mathrm{d}\tau'}{\mu}\\&=\mathbf{F}(\omega,\mu,\phi)\frac{\mu_0}{v^2(\mu_0-\mu')}\left\{\left[\left(\frac{\mu_0}{(\mu_0-\mu)}\right)\left(\exp\left(-\frac{v}{\mu_0}\tau\right)-\exp\left(-\frac{v}{\mu}\tau\right)\right)\right]\right.\\&\quad\left.-\left[\left(\frac{\mu'}{(\mu'-\mu)}\right)\left(\exp\left(-\frac{v}{\mu'}\tau\right)-\exp\left(-\frac{v}{\mu}\tau\right)\right)\right]\right\},\end{aligned} \tag{21}$$

$$\mathbf{I}_{\mathrm{d}}^{(2f-)}(\omega,\tau,\mu,\phi)=\int\int_{-1<\mu'<0}\mathbf{S}(\mu,\phi,\mu',\phi')\mathbf{I}_{\mathrm{d}}^{(2ba)}(\omega,\tau,\mu,\mu',\phi)\,\mathrm{d}\phi'\,\mathrm{d}\mu', \tag{22}$$

$$\begin{aligned}\mathbf{I}_{\mathrm{d}}^{(2ba)}(\omega,\tau,\mu,\mu',\phi)&=\int_0^{\tau}\exp\left(-\frac{v}{\mu}(\tau-\tau')\right)\mathbf{I}_{\mathrm{d}}^{(1+)}(\omega,\tau,\mu',\phi)\,\frac{\mathrm{d}\tau'}{\mu}\\&=\mathbf{F}(\omega,\mu,\phi)\frac{\mu_0}{v^2(\mu_0-\mu')}\left\{\left[\left(\frac{\mu_0}{(\mu_0-\mu)}\right)\left(\exp\left(-\frac{v}{\mu_0}\tau\right)-\exp\left(-\frac{v}{\mu}\tau\right)\right)\right]\right.\\&\quad\left.-\left[\left(\frac{\mu'}{(\mu'-\mu)}\right)\left(\exp\left(-\frac{v}{\mu'}\tau\right)-\exp\left(-\frac{v}{\mu}\tau\right)\right)\exp\left(\left(\frac{1}{\mu'}-\frac{1}{\mu_0}\right)\tau_0 v\right)\right]\right\}.\end{aligned} \tag{23}$$

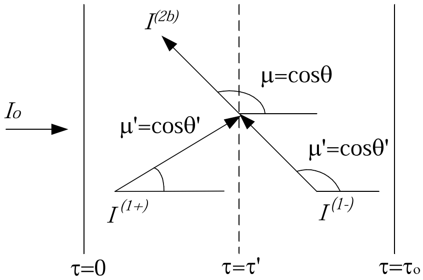

Fig. 2. The second-order scattering schematic.

In the same fashion, the second-order scattering solution for backward direction is

$$\mathbf{I}_{\rm d}^{(2b)}(\omega,\tau,\mu,\phi) = \mathbf{I}_{\rm d}^{(2b+)}(\omega,\tau,\mu,\phi) + \mathbf{I}_{\rm d}^{(2b-)}(\omega,\tau,\mu,\phi), \tag{24}$$

where $\mathbf{I}_{\rm d}^{(2b+)}$ is due to $\mathbf{I}_{\rm d}^{(1+)}$ and $\mathbf{I}_{\rm d}^{(2b-)}$ is due to $\mathbf{I}_{\rm d}^{(1-)}$

$$\mathbf{I}_{\rm d}^{(2b+)}(\omega,\tau,\mu,\phi) = \int\int_{-1<\mu'<0} \mathbf{S}(\mu,\phi,\mu',\phi')\mathbf{I}_{\rm d}^{(2fb)}(\omega,\tau,\mu,\mu',\phi)\,{\rm d}\phi'\,{\rm d}\mu', \tag{25}$$

$$\begin{aligned}\mathbf{I}_{\rm d}^{(2fb)}(\omega,\tau,\mu,\mu',\phi) &= \int_{\tau}^{\tau_0} \exp\left(-\frac{v}{\mu}(\tau-\tau')\right)\mathbf{I}_{\rm d}^{(1+)}(\omega,\tau,\mu',\phi)\,\frac{{\rm d}\tau'}{\mu} \\ &= \mathbf{F}(\omega,\mu,\phi)\frac{\mu_0}{v^2(\mu_0-\mu')}\left\{\left[\left(\frac{\mu_0}{(\mu_0-\mu)}\right)\left(\exp\left(\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)v\tau_0\right)\right.\right.\right. \\ &\quad \left.\left.-\exp\left(\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)v\tau\right)\right)\right] - \left[\left(\frac{\mu'}{(\mu'-\mu)}\right)\left(\exp\left(\left(\frac{1}{\mu}-\frac{1}{\mu'}\right)v\tau_0\right)\right.\right. \\ &\quad \left.\left.\left.-\exp\left(\left(\frac{1}{\mu}-\frac{1}{\mu'}\right)v\tau\right)\right)\right]\right\},\end{aligned} \tag{26}$$

$$\mathbf{I}_{\rm d}^{(2b-)}(\omega,\tau,\mu,\phi) = \int\int_{-1<\mu'<0} \mathbf{S}(\mu,\phi,\mu',\phi')\mathbf{I}_{\rm d}^{(2bb)}(\omega,\tau,\mu',\phi)\,{\rm d}\phi'\,{\rm d}\mu', \tag{27}$$

$$\begin{aligned}\mathbf{I}_{\rm d}^{(2bb)}(\omega,\tau,\mu,\mu',\phi) &= \int_{\tau}^{\tau_0} \exp\left(-\frac{v}{\mu}(\tau-\tau')\right)\mathbf{I}^{(1-)}(\omega,\tau,\mu',\phi)\,\frac{{\rm d}\tau'}{\mu} \\ &= \mathbf{F}(\omega,\mu,\phi)\frac{\mu_0}{v^2(\mu_0-\mu')}\left\{\left[\left(\frac{\mu_0}{(\mu_0-\mu)}\right)\left(\exp\left(\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)v\tau_0\right)\right.\right.\right. \\ &\quad \left.\left.-\exp\left(\left(\frac{1}{\mu_0}-\frac{1}{\mu}\right)v\tau\right)\right)\right] - \left[\left(\frac{\mu'}{(\mu'-\mu)}\right)\left(\exp\left(\left(\frac{1}{\mu}-\frac{1}{\mu_0}\right)v\tau_0\right)\right.\right. \\ &\quad \left.\left.\left.-\exp\left(\left(\frac{1}{\mu'}-\frac{1}{\mu_0}\right)v\tau_0\left(\frac{1}{\mu}-\frac{1}{\mu'}\right)v\tau\right)\right)\right]\right\}.\end{aligned} \tag{28}$$

In order to obtain the second-order scattering approximation solution, numerical integrations are required for solving Eqs. (20), (22), (25), and (27). There is a trade-off between accuracy and computational resources in this process. However, the computational resources required in this approximation are significantly less than those of the complete radiative transfer equation solution because we do not have to compute the eigen system type of equations.

3. Linear and circular polarization of the incident wave

From the previous section, we obtain solutions of the first-order and the second-order scattering approximations. They can be further simplified when Mie scattering is applied [5]. We explain the development in linear and circular polarization cases separately.

3.1. Linear polarization of the incident wave

The reduced specific intensity $\mathbf{I}_{\mathrm{ri}}$ for a linearly polarized wave in the x-direction is given by

$$\mathbf{I}_{\mathrm{ri}}(\omega,\tau,\mu,\phi) = I_0(\omega)[1 \quad 0 \quad 0 \quad 0]^{\mathrm{T}} \exp(-\tau)\delta(\mu-1)\delta(\phi), \tag{29}$$

where

$$I_0(\omega) = \begin{cases} I_0 & \text{for delta function pulse,} \\ I_0 \exp(-\omega^2 T_0^2/4) & \text{for Gaussian pulse of width } T_0, \end{cases} \tag{30}$$

where I_0 is the total incident intensity. In all our calculations, it is assumed to be 1. The diffuse specific intensity $\mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu,\phi)$ can be decomposed into Fourier expansions as

$$\mathbf{I}_{\mathrm{d}}(\omega,\tau,\mu,\phi) = [\mathbf{I}_{\mathrm{d}}]_{m=0}(\omega,\tau,\mu) + \left[\sum_{m=1}^{\infty}[\mathbf{I}_{\mathrm{dc}}]_m(\omega,\tau,\mu)\cos(m\phi) + \sum_{m=1}^{\infty}[\mathbf{I}_{\mathrm{ds}}]_m(\omega,\tau,\mu)\sin(m\phi)\right]. \tag{31}$$

For a normally incident plane wave ($\mu_0 = 1$), this expansion is reduced to two non-zero modes, which are, mode zero and mode two. Then, the specific intensity $\mathbf{I}_{\mathrm{d}}$ is reduced to

$$[\mathbf{I}_{\mathrm{d}}]_0(\omega,\tau,\mu) = [I_1 \quad I_2]^{\mathrm{T}}, \tag{32a}$$

$$[\mathbf{I}_{\mathrm{dc}}]_2(\omega,\tau,\mu) = [I_{1\mathrm{dc}} \quad I_{2\mathrm{dc}} \quad 0 \quad 0]^{\mathrm{T}}, \tag{32b}$$

$$[\mathbf{I}_{\mathrm{ds}}]_2(\omega,\tau,\mu) = [0 \quad 0 \quad U_{\mathrm{ds}} \quad V_{\mathrm{ds}}]^{\mathrm{T}}, \tag{32c}$$

where m corresponds to the mode number of the Fourier series. $m = 0$ is for mode zero, $m = 2$ is for mode two. With this expansion, the first-order scattering solution in forward direction from Eq. (17) becomes

$$\left[\mathbf{I}_{\mathrm{d}}^{(1+)}\right]_m(\omega,\tau,\mu) = [\mathbf{F}]_m(\omega,\mu)\frac{\mu_0}{v(\mu_0-\mu)}\left[\exp\left(-\frac{v}{\mu_0}\tau\right) - \exp\left(-\frac{v}{\mu}\tau\right)\right], \tag{33}$$

where $[\mathbf{I}_{\mathrm{d}}^{(1+)}]_0\,(\omega,\tau,\mu)$ is the solution to mode zero, $[\mathbf{I}_{\mathrm{d}}^{(1+)}]_2\,(\omega,\tau,\mu)$ is the solution to mode two, and

$$[\mathbf{F}]_m(\omega,\mu) = \begin{cases} \frac{I_0(\omega)}{2\sigma_{\mathrm{t}}}\begin{bmatrix} |A_{ll}(\mu)| \\ |A_{rr}(\mu)| \end{bmatrix} & \text{for } m = 0, \\ \frac{I_0(\omega)}{2\sigma_{\mathrm{t}}}\begin{bmatrix} |A_{ll}(\mu)| \\ -|A_{rr}(\mu)| \\ -2\Re|A_{ll}(\mu)A_{rr}^*(\mu)| \\ -2\Im|A_{ll}(\mu)A_{rr}^*(\mu)| \end{bmatrix} & \text{for } m = 2, \end{cases} \tag{34}$$

where the functions A_{ll} and A_{rr} are explained in Appendix A. In the same fashion, the first-order solution in backward direction in Eq. (18) and the second-order solution in Eqs. (19)–(28) for linearly polarized light can be calculated by substituting $[\mathbf{I}]_m$ for $\mathbf{I}$, $[\mathbf{F}]_m$ for $\mathbf{F}$, and $[\mathbf{S}]_m$ for $\mathbf{S}$, where

$$[\mathbf{S}]_m = \begin{cases} \mathbf{S}_1 & \text{for } m = 0, \\ \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{S}_3 & \mathbf{S}_4 \end{bmatrix} & \text{for } m = 2. \end{cases} \tag{35}$$

We can further simplify the solution by integration of $\mathbf{S}$ over ϕ as

$$\mathbf{L}_1(\mu,\mu') = \int_0^{2\pi} \mathbf{S}_1(\mu,\mu',\phi'-\phi)\cos(2(\phi'-\phi))\,\mathrm{d}(\phi'-\phi), \tag{36a}$$

$$\mathbf{L}_2(\mu,\mu') = \int_0^{2\pi} \mathbf{S}_2(\mu,\mu',\phi'-\phi)\sin(2(\phi'-\phi))\,\mathrm{d}(\phi'-\phi), \tag{36b}$$

$$\mathbf{L}_3(\mu,\mu') = \int_0^{2\pi} \mathbf{S}_3(\mu,\mu',\phi'-\phi)\sin(2(\phi'-\phi))\,\mathrm{d}(\phi'-\phi), \tag{36c}$$

$$\mathbf{L}_4(\mu,\mu') = \int_0^{2\pi} \mathbf{S}_4(\mu,\mu',\phi'-\phi)\cos(2(\phi'-\phi))\,\mathrm{d}(\phi'-\phi), \tag{36d}$$

and we define

$$[\mathbf{L}]_m = \begin{cases} \mathbf{L}_1 & \text{for } m = 0, \\ \begin{bmatrix} \mathbf{L}_1 & \mathbf{L}_2 \\ \mathbf{L}_3 & \mathbf{L}_4 \end{bmatrix} & \text{for } m = 2. \end{cases} \tag{37}$$

Then, the second-order solutions from Eqs. (20), (22), (25), and (27) become

$$\left[\mathbf{I}^{(2f+)}\right]_m(\omega,\tau,\mu) = \int_0^1 [\mathbf{L}]_m(\mu,\mu')\mathbf{I}^{(2fa)}(\omega,\tau,\mu,\mu')\,\mathrm{d}\mu', \tag{38a}$$

$$\left[\mathbf{I}^{(2f-)}\right]_m(\omega,\tau,\mu) = \int_{-1}^0 [\mathbf{L}]_m(\mu,\mu')\mathbf{I}^{(2fb)}(\omega,\tau,\mu,\mu')\,\mathrm{d}\mu', \tag{38b}$$

$$\left[\mathbf{I}^{(2b+)}\right]_m(\omega,\tau,\mu) = \int_0^1 [\mathbf{L}]_m(\mu,\mu')\mathbf{I}^{(2ba)}(\omega,\tau,\mu,\mu')\,\mathrm{d}\mu', \tag{38c}$$

$$\left[\mathbf{I}^{(2b-)}\right]_m(\omega,\tau,\mu) = \int_0^1 [\mathbf{L}]_m(\mu,\mu')\mathbf{I}^{(2bb)}(\omega,\tau,\mu,\mu')\,\mathrm{d}\mu'. \tag{38d}$$

In linear polarization, the co-polarized component is I_1 and the cross-polarized component is I_2.

3.2. Circular polarization of the incident wave

The reduced intensity $\mathbf{I}_{\mathrm{ri}}$ for a left-handed circularly polarized wave is given by

$$\mathbf{I}_{\mathrm{ri}}(\omega,\tau,\mu,\phi) = I_0(\omega)[1/2 \quad 1/2 \quad 0 \quad 1]^{\mathrm{T}}\exp(-\tau)\delta(\mu-1)\delta(\phi), \tag{39}$$

where $I_0(\omega)$ obeys Eq. (30). It can also be expanded using Fourier series expansion in ϕ dependence in the same way as the linear polarized wave shown in Eq. (31). For a normally incident wave, the expansion reduces to just a single zero mode. In this mode, the equation can be uncoupled into two separate equations, which are more numerically efficient to solve. Then, equivalent to Eq. (32a) for linear polarization, the specific intensity $\mathbf{I}_{\mathrm{d}}$ is reduced to

$$[\mathbf{I}_{\mathrm{d}}]_n(\omega,\tau,\mu) = \begin{cases} [I_1 \quad I_2]^{\mathrm{T}} & \text{for } n=1, \\ [U \quad V]^{\mathrm{T}} & \text{for } n=2, \end{cases} \tag{40}$$

where n corresponds to the uncoupled equation number, $n = 1$ is for the equation of I_1 and I_2 in mode zero, $n = 2$ is for the equation of U and V in mode zero. Thus, the first-order scattering approximation solution from Eq. (17) becomes

$$\left[\mathbf{I}_{\mathrm{d}}^{(1+)}\right]_n(\omega,\tau,\mu) = [\mathbf{F}]_n(\omega,\mu)\frac{\mu_0}{v(\mu_0-\mu)}\left[\exp\left(-\frac{v}{\mu_0}\tau\right) - \exp\left(-\frac{v}{\mu}\tau\right)\right], \tag{41}$$

where $[\mathbf{I}_{\mathrm{d}}^{(1+)}]_1(\omega,\tau,\mu)$ is the solution to the equation of I_1 and I_2 in mode zero , $[\mathbf{I}_{\mathrm{d}}^{(1+)}]_2(\omega,\tau,\mu)$ is the solution to the equation of U and V in mode zero,

$$[\mathbf{F}]_n(\omega,\mu) = \begin{cases} \frac{I_0(\omega)}{2\sigma_{\mathrm{t}}}\begin{bmatrix} |A_{ll}(\mu)| \\ |A_{rr}(\mu)| \end{bmatrix} & \text{for } n=1, \\ \frac{I_0(\omega)}{\sigma_{\mathrm{t}}}\begin{bmatrix} \Im|A_{ll}(\mu)A_{rr}^*(\mu)| \\ \Re|A_{ll}(\mu)A_{rr}^*(\mu)| \end{bmatrix} & \text{for } n=2, \end{cases} \tag{42}$$

and

$$[\mathbf{S}]_n = \begin{cases} \mathbf{S}_1 & \text{for } n=1, \\ \mathbf{S}_4 & \text{for } n=2. \end{cases} \tag{43}$$

In the same fashion as in the linearly polarized wave case, we find the solution to second-order scattering approximation for circular polarization case to be

$$\left[\mathbf{I}^{(2f+)}\right]_n(\omega,\tau,\mu) = \int_0^1 [\mathbf{L}]_n(\mu,\mu')\mathbf{I}^{(2fa)}(\omega,\tau,\mu,\mu')\,\mathrm{d}\mu', \tag{44a}$$

$$\left[\mathbf{I}^{(2f-)}\right]_n(\omega,\tau,\mu) = \int_{-1}^0 [\mathbf{L}]_n(\mu,\mu')\mathbf{I}^{(2fb)}(\omega,\tau,\mu,\mu')\,\mathrm{d}\mu', \tag{44b}$$

$$\left[\mathbf{I}^{(2b+)}\right]_n(\omega,\tau,\mu) = \int_0^1 [\mathbf{L}]_n(\mu,\mu')\mathbf{I}^{(2ba)}(\omega,\tau,\mu,\mu')\,\mathrm{d}\mu', \tag{44c}$$

$$\left[\mathbf{I}^{(2b-)}\right]_n(\omega,\tau,\mu) = \int_0^1 [\mathbf{L}]_n(\mu,\mu')\mathbf{I}^{(2bb)}(\omega,\tau,\mu,\mu')\,\mathrm{d}\mu', \tag{44d}$$

where

$$[\mathbf{L}]_n = \begin{cases} \mathbf{L}_1 & \text{for } n=1, \\ \mathbf{L}_4 & \text{for } n=2. \end{cases} \tag{45}$$

In circular polarization, the co-polarized component is $(I1 + I2 + V)/2$ and the cross-polarization component is $(I1 + I2 - V)/2$.

4. Numerical results

We apply the first-order and the second-order scattering approximations to a problem of a 1-μm-wavelength light pulse propagating through a random medium. The random medium contains dielectric spheres in the homogeneous background. The dielectric spheres have a Gaussian size distribution with average diameter of 10 μm. We make some comparisons between the first-order approximation, the second-order approximation and the radiative transfer equation solution obtained from discrete ordinates method with the same conditions.

First, we consider the angular behavior of the intensity. Fig. 3 shows the co-polarized components of the angular spectrum of linearly and circularly polarized waves at an optical depth of 1. The results suggest that both the first-order approximation and the second-order approximation are relatively accurate near the optical axis. When the angle of observation gets further from the axis, the second-order shows better accuracy than the first-order. This is expected since the wider the angle, the more the effect of multiple scattering. The results also suggest that linear and circular polarizations have the same angular behaviors.

Next, we investigate the effect of the optical depth on the approximation. The angular spectrum comparisons for linear polarization are shown in Fig. 4. The results indicate that, as the optical depth gets larger, the second-order approximation loses accuracy when compared with the complete radiative transfer

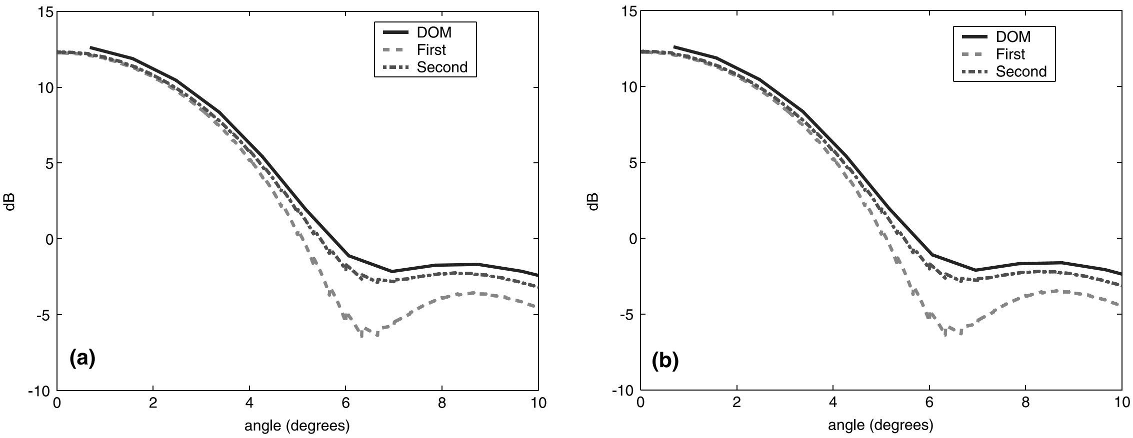

Fig. 3. Co-polarized component angular spectrum of comparison for the first-order (First), the second-order (Second) approximation compared with the Discrete Ordinates Method (DOM) at the Optical Depth (OD) of 1: (a) linear polarization, and (b) circular polarization.

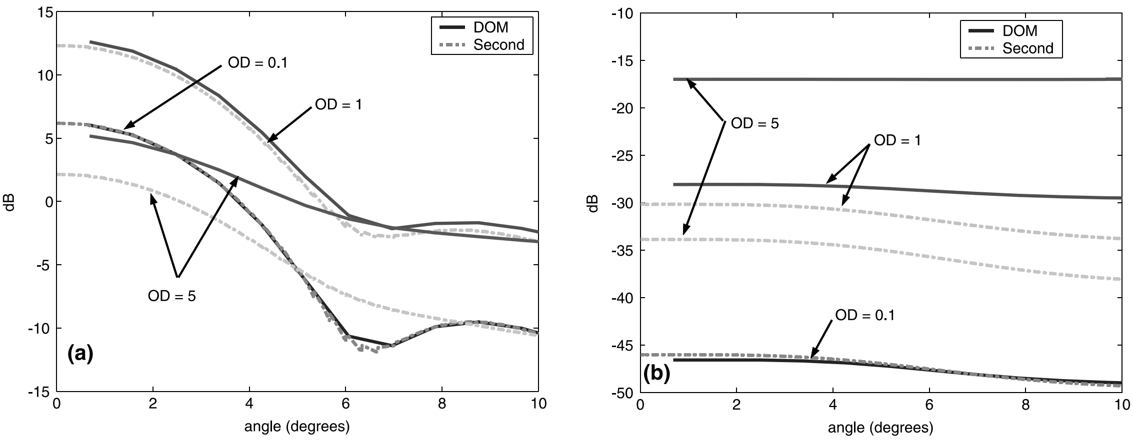

Fig. 4. Linear polarization angular spectrum of the second-order approximation (Second) compared with the Discrete Ordinates Method (DOM) at different Optical Depth: (a) co-polarization, and (b) cross-polarization.

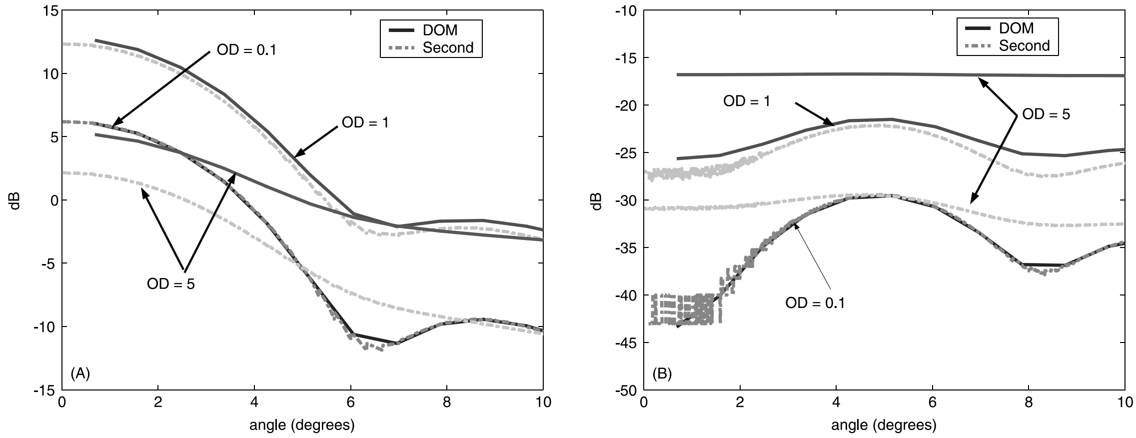

Fig. 5. Circular polarization angular spectrum of the second-order approximation (F + S) compared with the Discrete Ordinates Method (DOM) at different Optical Depth: (a) co-polarization, and (b) cross-polarization.

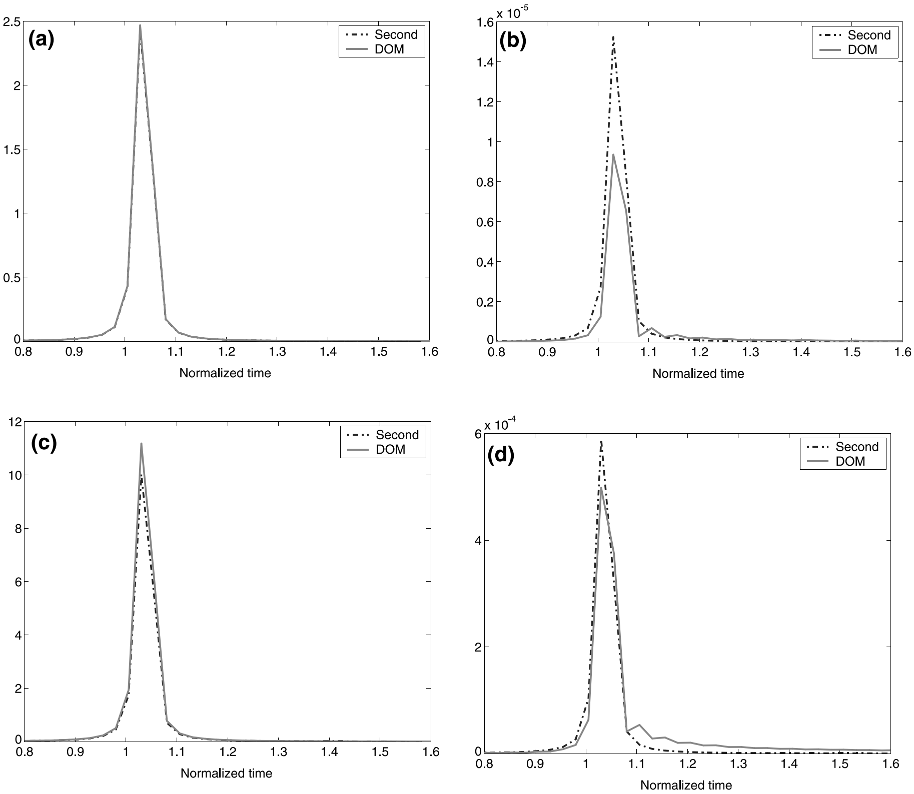

Fig. 6. Linear polarization time-domain of the second-order (Second) compared with the Discrete Ordinates Method (DOM): (a) co-polarization at OD = 0.1, (b) cross-polarization at OD = 0.1, (c) co-polarization at OD = 1, and (d) cross-polarization at OD = 1.

solution. This is due to the fact that high-multiple scattering events increase as the optical depth gets large. The second-order approximation is based on the combination of one and two scattering events only. The accuracy of the approximation for the co-polarized component is better relative to the cross-polarized component. That is also because the cross-polarized component is believed to have already encountered many scattering events. Thus, more scattering orders are needed to capture the cross-polarization behaviors. Fig. 5 shows the same comparison as Fig. 4 for circular polarization. The co-polarized component exhibits the same behavior as those of linear polarization. The cross-polarization shows considerable variation at small angles close to the optical axis. This might be due to the fact that the approximation of circular cross-polarized component comes from subtraction of two approximately equal numbers, which are $(I1 + I2)$ and (V). Through the process of approximation, those numbers carry some errors already, and upon subtraction, the effect of the errors is magnified.

Finally, we calculate the pulse-wave cases. The results are exhibited in a series of figures showing amplitude of co-polarized and cross-polarized components as a function of normalized time. Fig. 6 shows the linear polarization for optical depths of 0.1 and 1. Fig. 7 shows the same for optical depths of 2 and 5. There are some noticeable amplitude discrepancies in the cross-polarization results. The cross-polarized component is created by many scattering events. Therefore, the approximations based on only one and two scattering events may not be accurate. The pulse shape of linear co-polarized component using first-order and second-order approximations in the case of optical depth of 0.1 coincide with that of discrete ordinates method showing good accuracy. However, in the case of optical depth of 1, there are a small amplitude

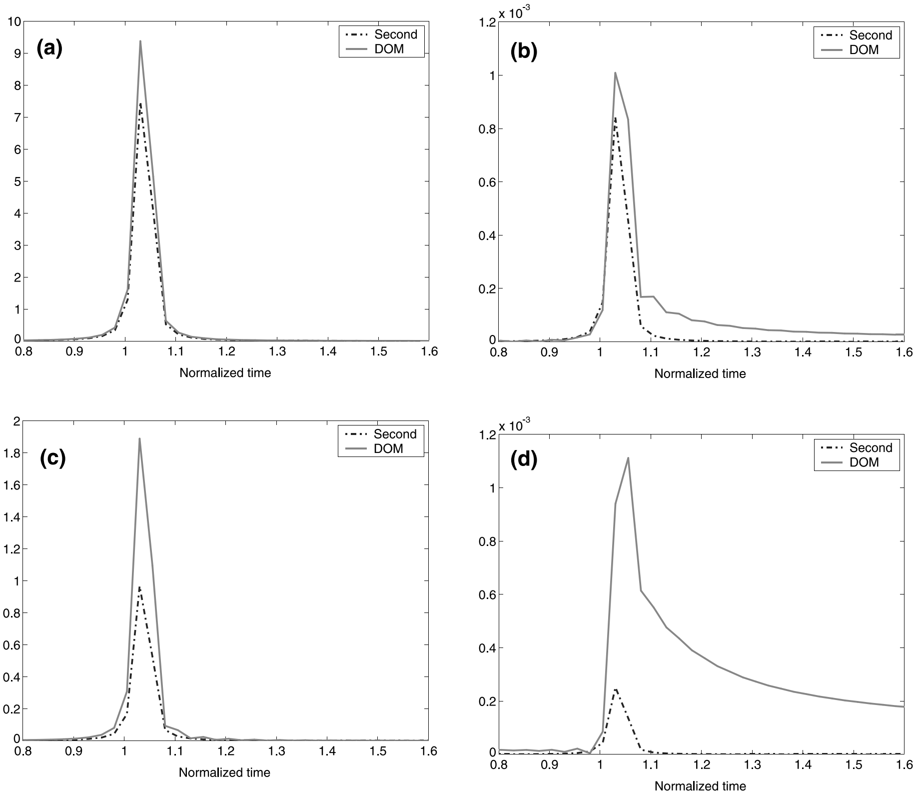

Fig. 7. Linear polarization time-domain of the second-order (Second) compared with the Discrete Ordinates Method (DOM): (a) co-polarization at OD = 2, (b) cross-polarization at OD = 2, (c) co-polarization at OD = 5, and (d) cross-polarization at OD = 5.

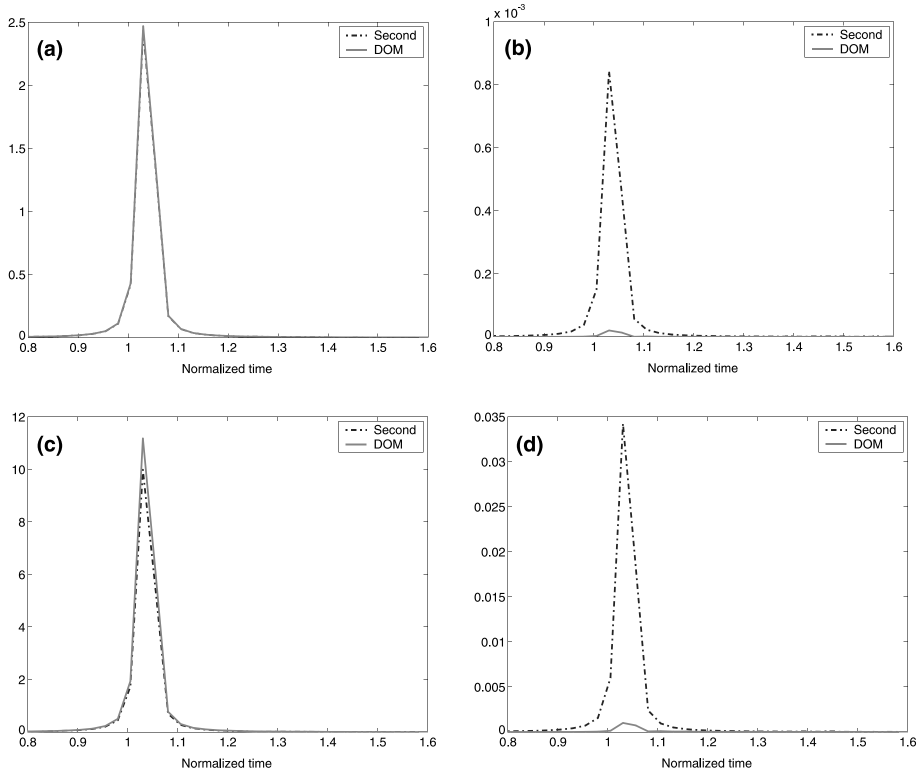

Fig. 8. Circular polarization time-domain of the second-order (Second) compared with the Discrete Ordinates Method (DOM): (a) co-polarization at OD = 0.1, (b) cross-polarization at OD = 0.1, (c) co-polarization at OD = 1, and (d) cross-polarization at OD = 1.

difference between the first-order and second-order results and the discrete ordinates method indicating that as the optical depth gets larger, the accuracy degrades. For the case of optical depth of 5, the amplitude of the co-polarized component using the first-order and second-order approximations is about half (3 dB) of that of the discrete ordinates method showing poor accuracy of approximations. It is to be expected because the optical depth is too high to be represented by only one and two scattering events. Also, there is a large time spread in the cross-polarization case using radiative transfer solution. The second-order results do not exhibit such a property because they are based on just one and two scattering events, which cannot cause much time spread. Fig. 8 shows the circular polarization for optical depths of 0.1 and 1, while Fig. 9 shows the same for optical depths of 2 and 5. They suggest the same conclusion as in the linear cases except that circular polarization appears to lose more accuracy in the cross-polarized component than that of linear polarization for the same reason as discussed previously.

5. Conclusions

We have derived here the first-order and the second-order scattering approximations from the pulse vector radiative transfer equation. The results suggest that they are accurate, compared to the full radiative transfer equation solution, only when the angle of observation is close to the optical axis and the optical depths are small. Also, the accuracy of approximation for the case of linear polarization is, in general, better than that for the case of circular polarization.

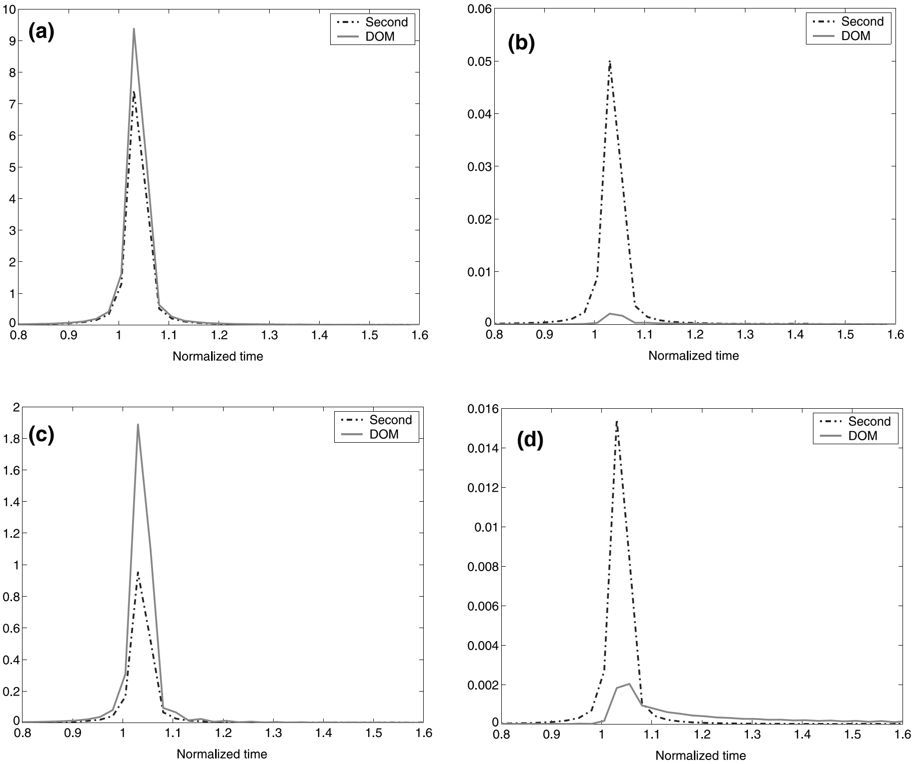

Fig. 9. Circular polarization time-domain of the second-order (Second) compared with the Discrete Ordinates Method (DOM): (a) co-polarization at OD = 2, (b) cross-polarization at OD = 2, (c) co-polarization at OD = 5, and (d) cross-polarization at OD = 5.

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Appendix A

The scattering amplitudes for spherical particles are given by

$$f_{11} = (l,l)T_1 + (r,r)T_2, \tag{A.1a}$$

$$f_{12} = -(r,l)T_1 + (l,r)T_2, \tag{A.1b}$$

$$f_{21} = -(l,r)T_1 + (r,l)T_2, \tag{A.1c}$$

$$f_{22} = (r,r)T_1 + (l,l)T_2, \tag{A.1d}$$

where

$$(l,l) = \left[(1-\mu^2)(1-\mu'^2)\right]^{1/2} + \mu\mu'\cos(\phi'-\phi), \tag{A.2a}$$

$$(l, r) = -\mu' \sin(\phi' - \phi), \tag{A.2b}$$

$$(r, l) = \mu' \sin(\phi' - \phi), \tag{A.2c}$$

$$(r, r) = \cos(\phi' - \phi), \tag{A.2d}$$

$$T_1(x) = \frac{A_{rr}(\chi) - \chi A_{ll}(\chi)}{1 - \chi^2}, \tag{A.2e}$$

$$T_2(x) = \frac{A_{ll}(\chi) - \chi A_{rr}(\chi)}{1 - \chi^2}, \tag{A.2f}$$

$$\chi = \cos\Theta = [(1 - \mu^2)(1 - \mu'^2)]^{1/2} \cos(\phi' - \phi) + \mu\mu', \tag{A.2g}$$

$$\mu = \cos(\theta), \quad \mu' = \cos(\theta') \tag{A.2h}$$

where (θ, ϕ) and (θ', ϕ') correspond to the incident and scattered wave directions, respectively. The functions A_{ll} and A_{rr} are related to the scattering function s_1 and s_2 for Mie solution explained by Van de Hulst [5] as

$$A_{ll} = \mathrm{i}s_2^*/k, \quad A_{rr} = \mathrm{i}s_1^*/k. \tag{A.3}$$

References

[1] A. Ishimaru, S. Jaruwatanadilok, Y. Kuga, Appl. Opt. 40 (2001) 5495.
[2] A.D. Kim, S. Jaruwatanadilok, A. Ishimaru, Y. Kuga, Proc. SPIE 4257 (2001) 90.
[3] A. Ishimaru, Wave Propagation and Scattering in Random Media, IEEE Press, New York, 1997.
[4] R.L.-T. Cheung, A. Ishimaru, Appl. Opt. 21 (1982) 3792.
[5] H.C. Van de Hulst, Light Scattering by Small Particles, Wiley, New York, 1957.
[6] Y. Kuga, A. Ishimaru, Q. Ma, Radio Sci. 24 (1989) 247.